Semiclassical approach to color confinement via compactifications

Yuya Tanizaki (Yukawa institute, Kyoto)

Goal of this talk

$\boxed{\text{Review a semiclassical approach to confinement of 4d YM on } \mathbb{R}^3\times S^1 \text{ and } \mathbb{R}^2\times T^2.}$

- Introduction

- Confinement on $\mathbb{R}^2\times T^2$ with 't Hooft flux via center vortices
  (Tanizaki, Ünsal 2022
  cf. van Baal '80s, Yamazaki Yonekura 2017, Cox, Poppitz, Wandler 2021 for $\mathbb{R}\times T^3$)

- Confinement on $\mathbb{R}^3\times S^1$ via monopoles & bions
  (Ünsal, Yaffe, Shifman, Poppitz, Anber, ... 2007–.
  cf. Davis, Hollowood, Khoze, Mattis 2000 for SYM)

- Unification of monopole & center vortex mechanisms.
  (Hayashi, Tanizaki 2024)

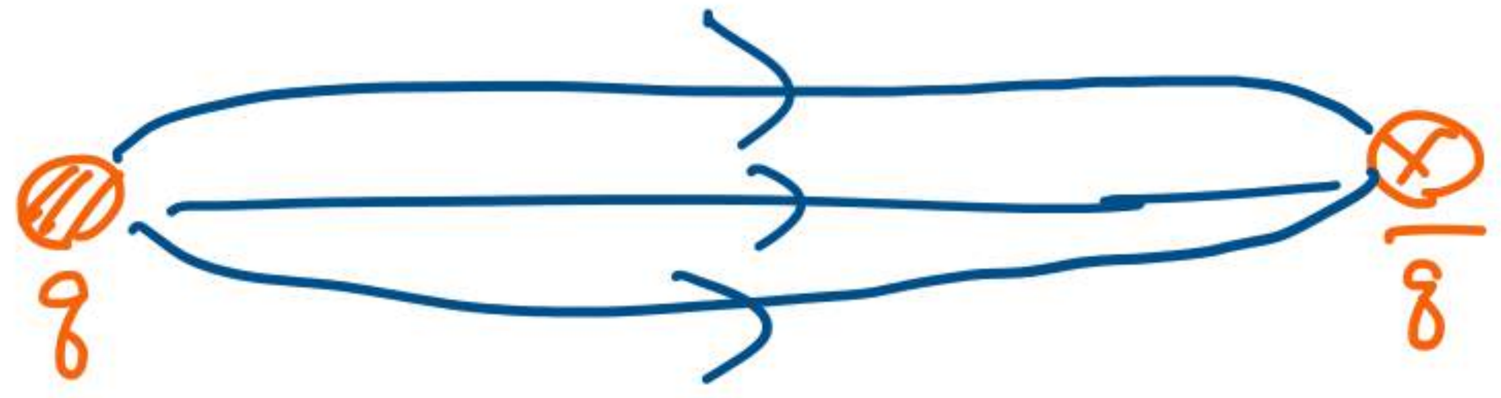
Some known facts of 4d YM

$$\mathcal{L} = \frac{1}{g^2}\,\mathrm{tr}(F\wedge *F) \; + i\,\frac{\theta}{8\pi^2}\,\mathrm{tr}(F\wedge F).$$

Theory has the typical energy scale

$$\Lambda \sim \mu\, e^{-\frac{8\pi^2}{\frac{11}{3}N g^2(\mu)}}.$$

Below this scale, the color charge is confined:



We would like to understand how the color confinement occurs.

Although 4d YM on $\mathbb{R}^4$ is tough to be solved,
there are its various deformations satisfying

- the deformed model is easy to solve analytically, and
- it maintains the color confinement.

Examples

× Lattice strong coupling (Wilson '74)

× Holography (Witten '98)

× Supersymmetric version (Seiberg, Witten '94)

YM on the compactified spacetime is another example.

Confinement & large-N volume independence

(* Our work for YM on $\mathbb{R}^2 \times T^2$ does <u>NOT</u> take the large-N limit.
Still, many ideas are imported from twisted EK by Gonzalez-Arroyo, Okawa.)

- The large-N limit is a kind of the classical limit $\hbar \to 0$:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle + O\left(\frac{1}{N}\right).$$

- $\boxed{\text{If the system is confined}}$ (i.e. $\langle W(S^1) \rangle = 0$),
then the single-trace observables do not depend on the volume.
(large-N volume independence) (Eguchi, Kawai '82)

- The naive one-plaquette model shows deconfinement in the large-N limit.
Gonzalez-Arroyo, Okawa ('83, '10): Introduce the $\boxed{\text{'t Hooft twist.}}$
(cf. Bietenholz, Nishimura, Susaki, Volkholz '06, Teper, Vairinhos '07, Azeyanagi, Hanada, Hirata, Ishikawa '08)

Confinement on $\mathbb{R}^2 \times T^2$ with 't Hooft flux via center vortices

Symmetry twisted BC = Flat background gauge field

Consider a complex scalar field

$$\mathcal{L} = |\partial_\mu \phi|^2 + V(|\phi|^2)$$

w/ U(1)-twisted b.c. $\phi(x_k + L) = e^{i\alpha_k}\phi(x_k)$.

$$\Updownarrow \qquad \phi(x) := e^{\sum_k i \frac{\alpha_k}{L} x_k} \tilde{\phi}(x_\mu)$$

$$\mathcal{L} = |(\partial_\mu + iA_\mu)\tilde{\phi}|^2 + V(|\tilde{\phi}|^2)$$

w/ the background gauge field $A = \sum_k \frac{\alpha_k}{L} dx_k$.

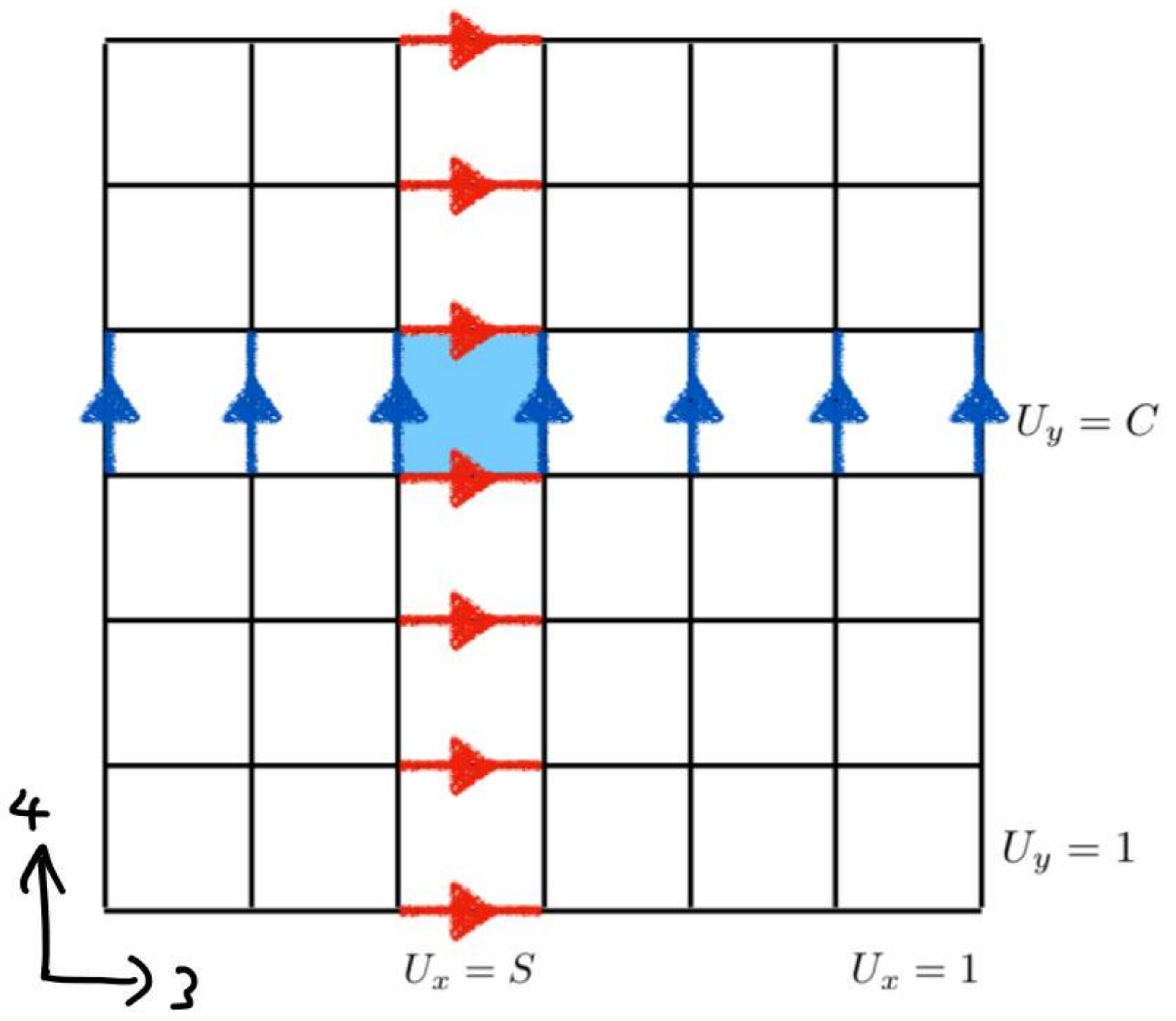
In general, a symmetry twisted boundary condition can be described as a flat background gauge field.

't Hooft twist = Background gauge field for $\mathbb{Z}_N^{(1)}$ symmetry.

Classical configuration (= twist eater) & center symmetry

Lattice action

$$S_W[U_\ell, B] = -\frac{1}{g^2}\sum_p \left(e^{-iB_p}\,\mathrm{tr}[U_p] + e^{iB_p}\,\mathrm{tr}[U_p^\dagger] \right).$$



$$B_p = \begin{cases} \frac{2\pi}{N} & \text{(for the plaquette indicated with light blue)} \\ 0 & \text{(otherwise)} \end{cases}$$

We can minimize this action by setting

$$U_\ell = \begin{cases} S = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \\ C = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \\ \mathbb{1} \end{cases}$$

$$\Rightarrow \quad P_3 = S \quad , \quad P_4 = C .$$

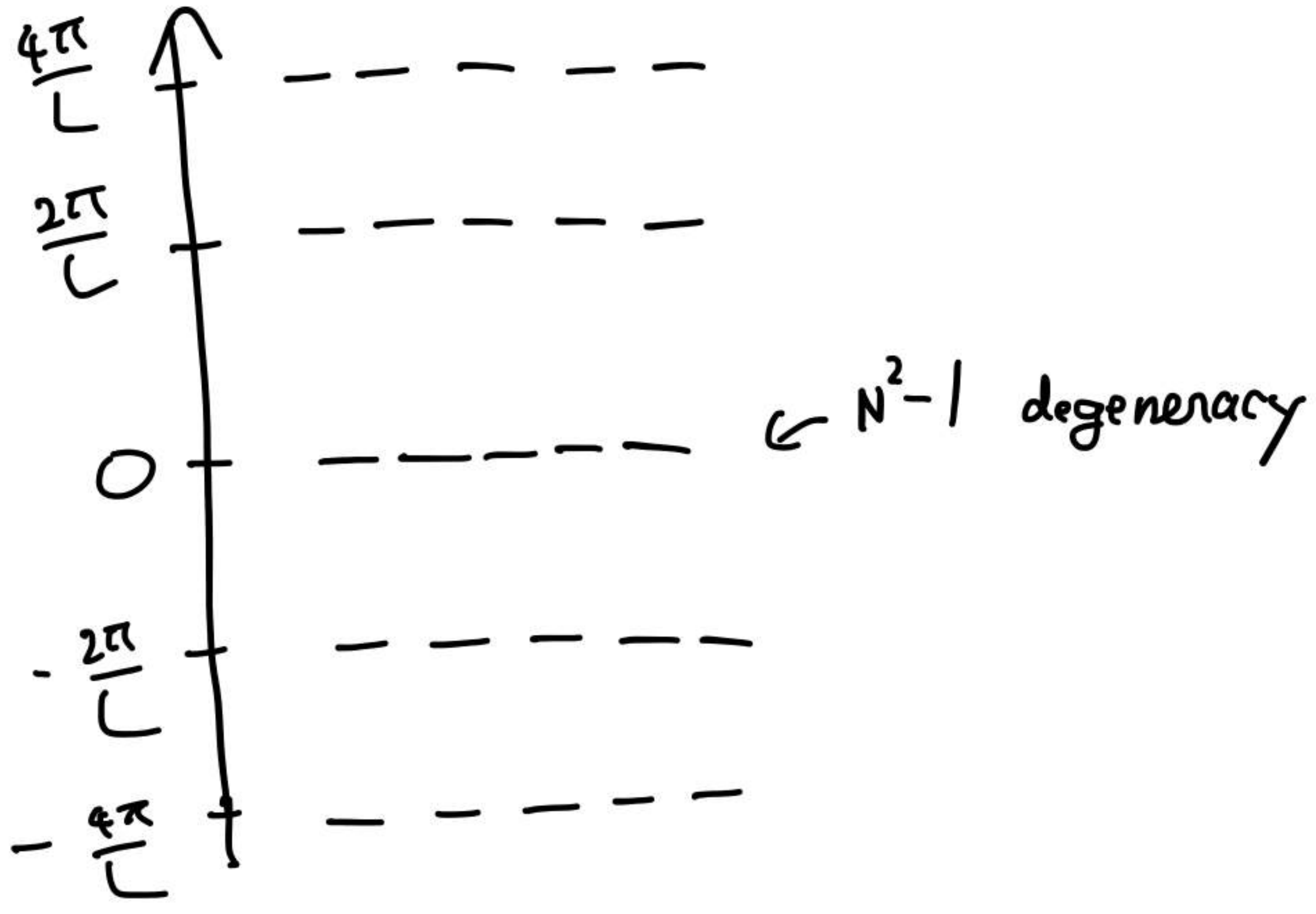
This configuration completely preserves $\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

Perturbative spectrum

Put 4d YM on $\mathbb{R}^2 \times T^2$.

Perturbative spectrum of 2d gauge fields on $\mathbb{R}^2$ part.

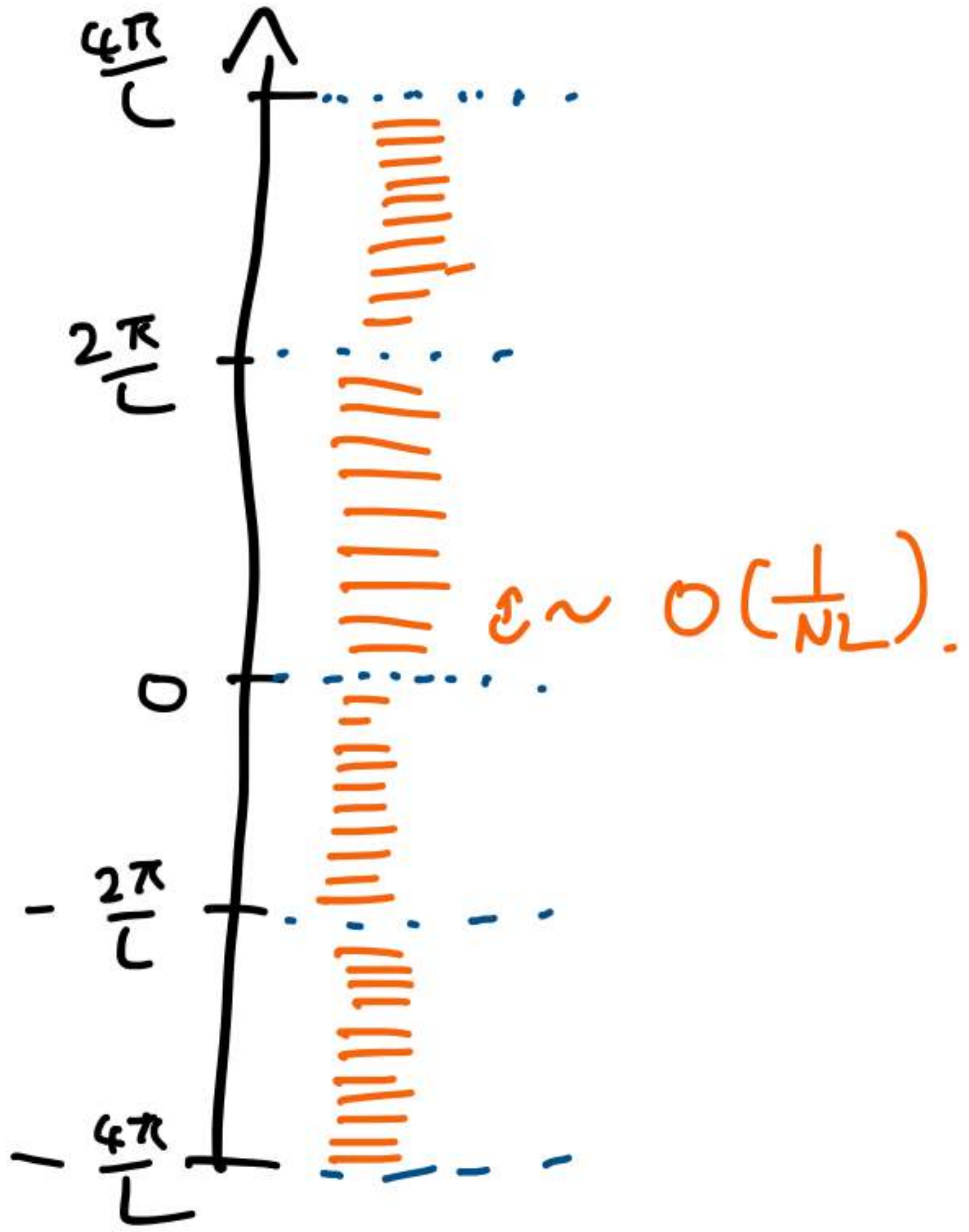
Without twist



$$M_{KK}^2 = \left(\frac{2\pi}{L}\right)^2 (l_3^2 + l_4^2)$$

- There exist zero modes.

With twist

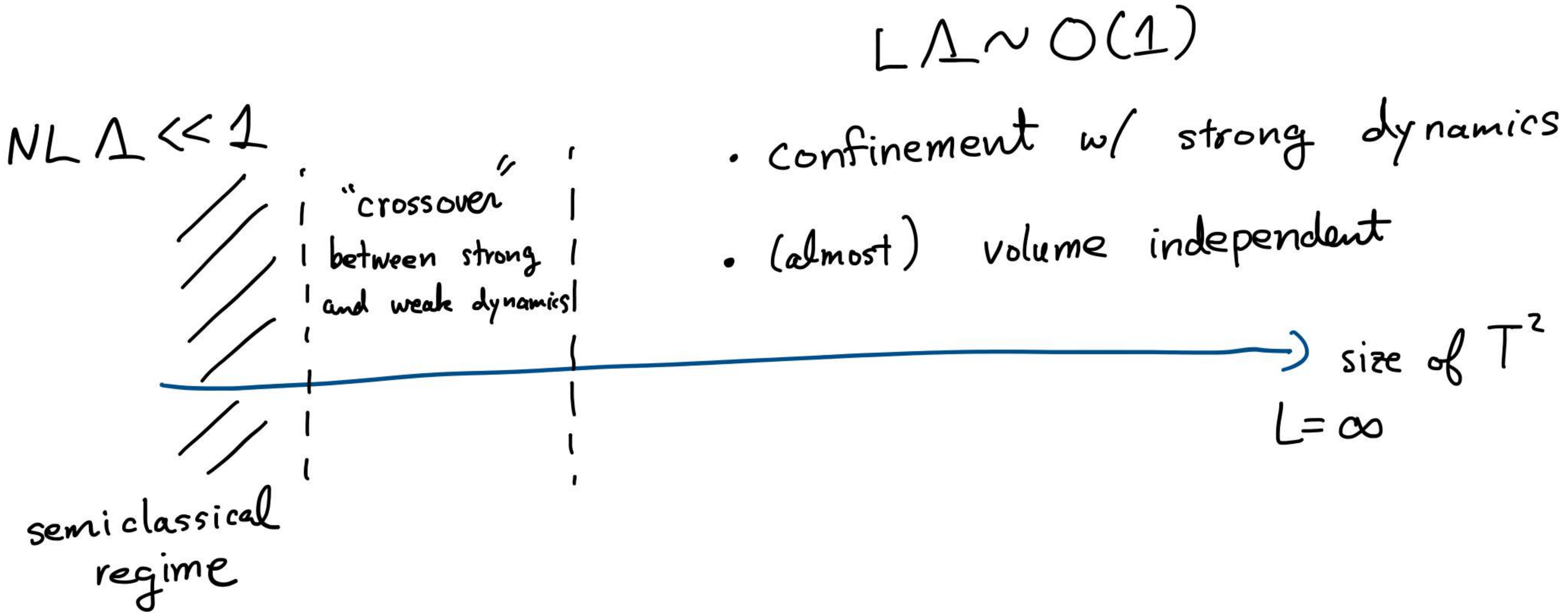


$$M_{KK} = \left(\frac{2\pi}{L}\right)^2 \left(\left(l_3 + \frac{p_3}{N}\right)^2 + \left(l_4 + \frac{p_4}{N}\right)^2 \right)$$

for the color basis $e^{-\frac{2\pi i}{N}\frac{p_3 p_4}{2}} C^{-p_3} S^{p_4}$.

- No zero modes.
- Gap $\sim \frac{2\pi}{NL}$.

Semiclassical regime for 4d YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux



We can prove confinement for $NL\Lambda \ll 1$ using the semiclassical method.

Perturbative analysis of $SU(N)$ YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

- $\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry is unbroken.

- 2d gluons are gapped.

  $\Leftarrow$ Polyakov loops along T^2 are adjoint Higgs fields for $\mathbb{R}^2$.

  $P_3 = S$, $P_4 = C$ gives

  $$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N.$$

  Weak-coupling analysis is free from IR divergences.

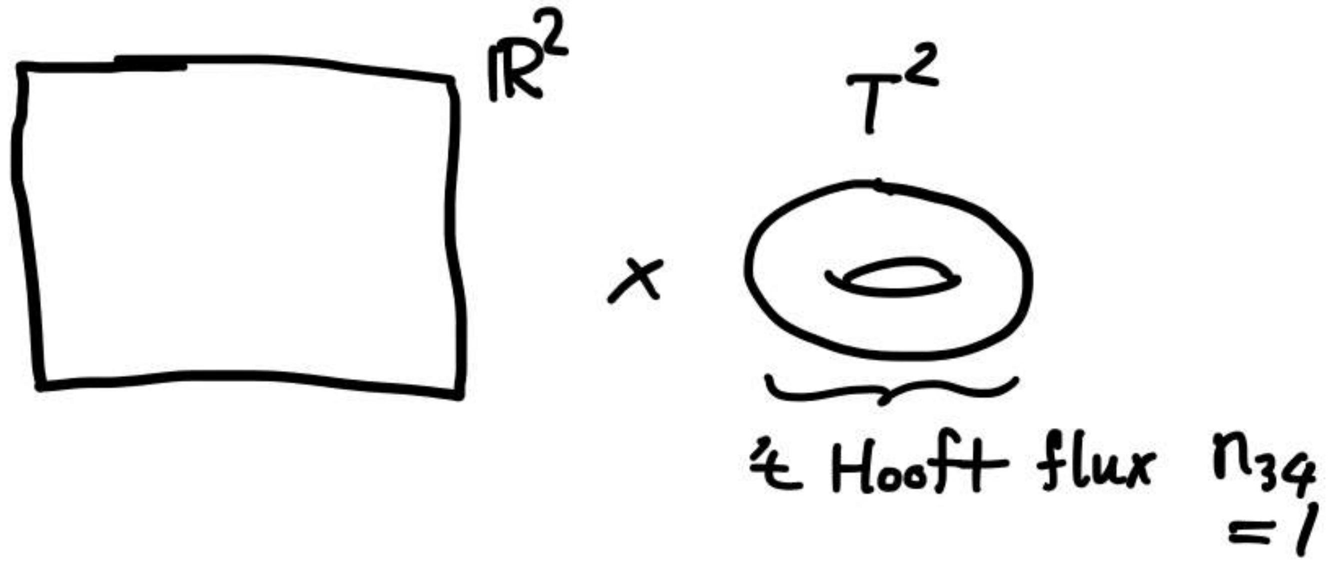
- However, Wilson loops inside $\mathbb{R}^2$ obey perimeter laws.

  $\Downarrow$

  This issue is resolved by the semiclassics with center vortices.

Center vortex as a fractional instanton on $\mathbb{R}^2 \times T^2$

In this setup, the minimal topological charge is given by



$$\boxed{Q_{top} = \frac{1}{8\pi^2}\int \mathrm{tr}(F \wedge F) = \frac{1}{N}}$$

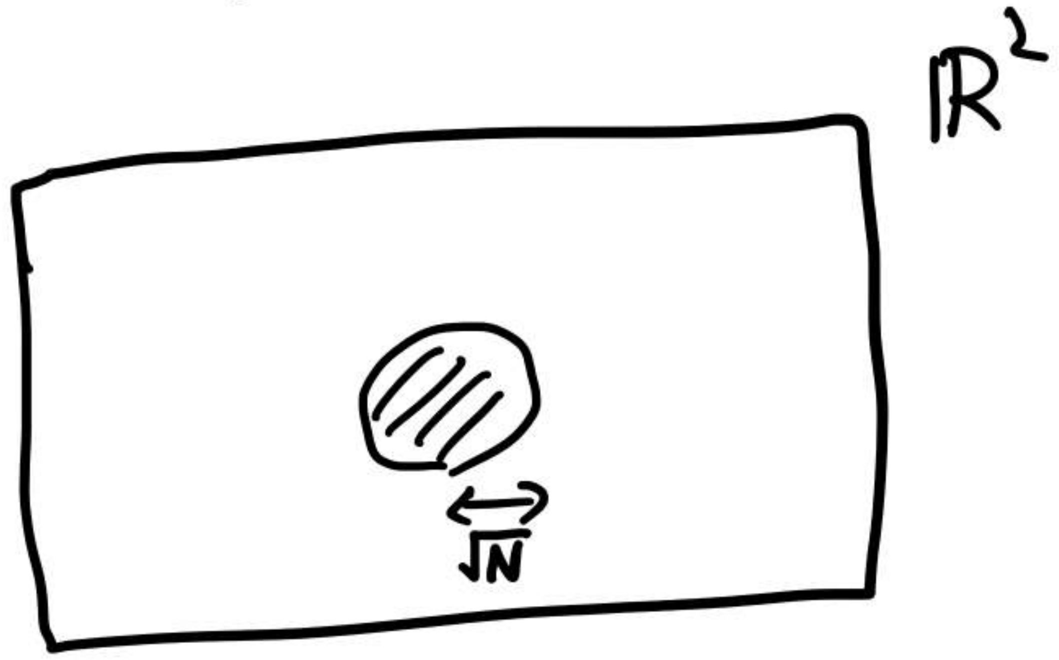
(More precisely, $Q_{top} \in \frac{1}{N}\left(\frac{-\varepsilon_{\mu\nu\rho\sigma} n_{\mu\nu} n_{\rho\sigma}}{8}\right) + \mathbb{Z}$ (van Baal '82))

If there exists a self-dual configuration, its Yang-Mills action becomes

$$\boxed{S_{YM} = \frac{8\pi^2}{g^2}|Q_{top}| = \frac{8\pi^2}{g^2 \cdot N}}$$

Gonzalez-Arroyo, Montero '98, Montero '99 numerically confirmed such a classical solution exists:

center vortex or fractional instanton.



$$\left(\begin{array}{l} Q_{top} = \frac{1}{N} \\ S_{YM} = \frac{8\pi^2}{g^2 N} \end{array}\right)$$

(cf. Garcia Perez, Gonzalez-Arroyo, '92, Itou '18)

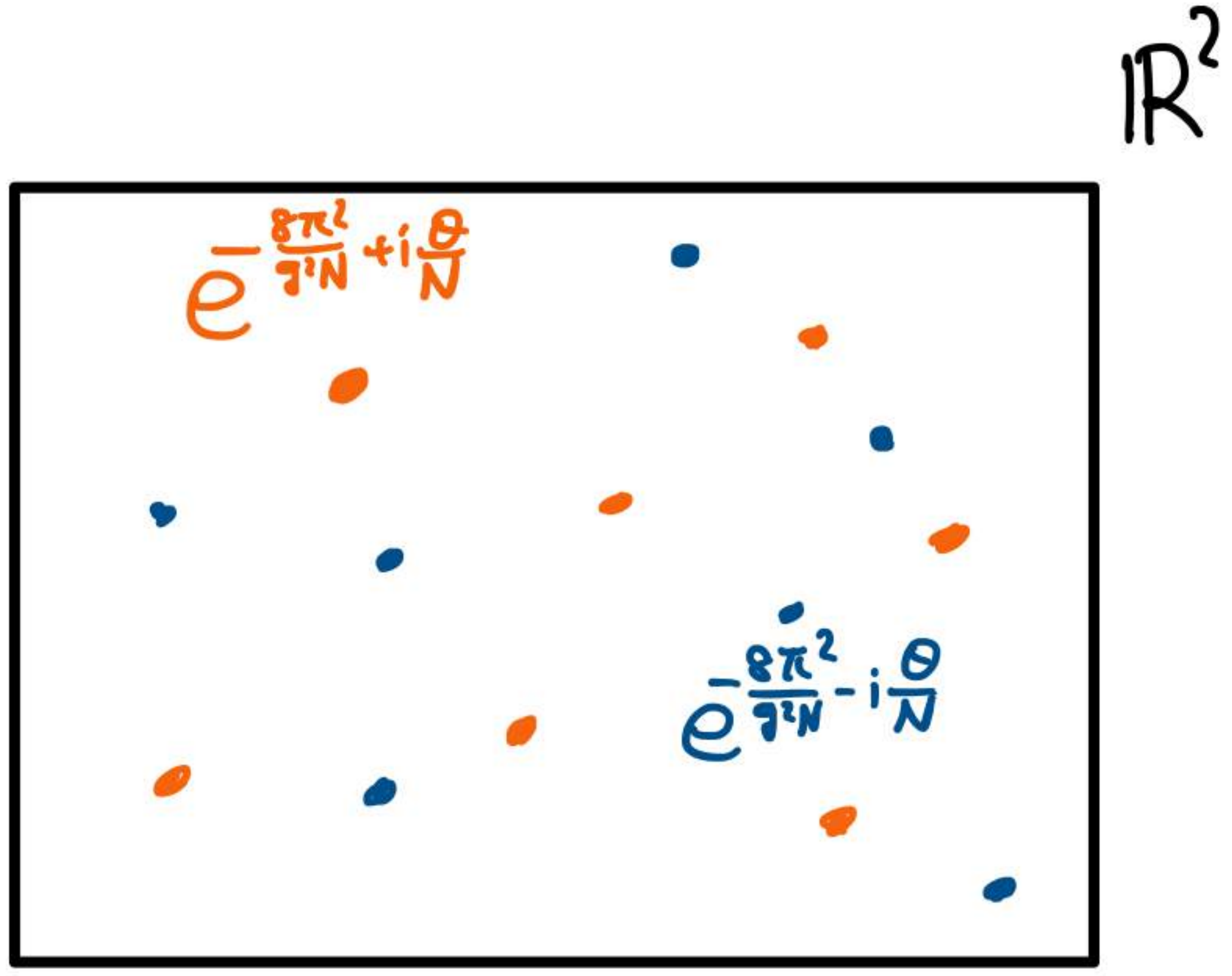
Dilute gas approximation

2d gluon fields are perturbatively gapped by 't Hooft twist.

$\Rightarrow$ Center vortex, or fractional instanton, does NOT have the size moduli.

$\Rightarrow$ Dilute gas approximation is available.

(* In 4d pure YM, DIGA is invalidated because of IR divergences.)



n : # of vortices

$\bar{n}$: # of anti-vortices

$$Q_{top} = \frac{n-\bar{n}}{N}.$$

Partition funcion on $\underset{\to \mathbb{R}^2}{M_2} \times T^2$ & θ-dependence

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

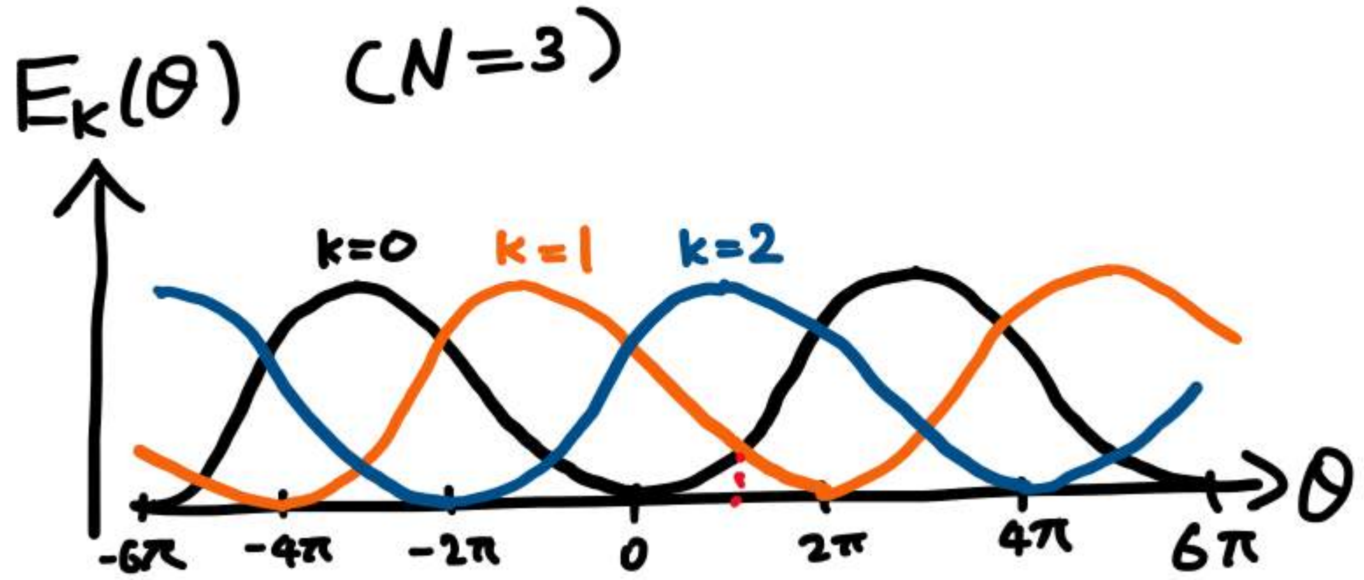
$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}}_{\text{vortex}}\Big)^{n} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}}_{\text{anti-vortex}}\Big)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\Big[-V\Big(\underbrace{-2K\, e^{-\frac{8\pi^2}{g^2N}} \cos\Big(\frac{\theta-2\pi k}{N}\Big)}_{E_k(\theta)\,:\ \text{Ground-state energy densities}}\Big)\Big]$$

$\Rightarrow$
- N-branch structure of ground states.
- Each branch has a fractional θ-dependence.



Confinement on $\mathbb{R}^3 \times S^1$ via monopoles & bions

Setup for semiclassics on $\mathbb{R}^3 \times S^1$

- 4d YM on $\mathbb{R}^3 \times S^1$ + Double-trace potential $\sum_{n=1}^{N-1} |\mathrm{tr}(P_4^n)|^2$.

  * Without the additional potential, thermal YM is deconfined.

  * One may regard the double-trace term as an effective potential of massive adjoint fermions w/ periodic b.c.

- $NL\Lambda \ll 1$.

The double-trace potential forces that $P_4 = C \propto \mathrm{diag}(1, \omega, \cdots, \omega^{N-1})$.

KK mass of $a_I^{ij} = \dfrac{2\pi}{NL}(i-j)$.

Below $E < \dfrac{2\pi}{NL}$, the low-energy gauge group is abelianized: $SU(N) \xrightarrow{\text{Higgs}} U(1)^{N-1}$.

3d Abelian duality

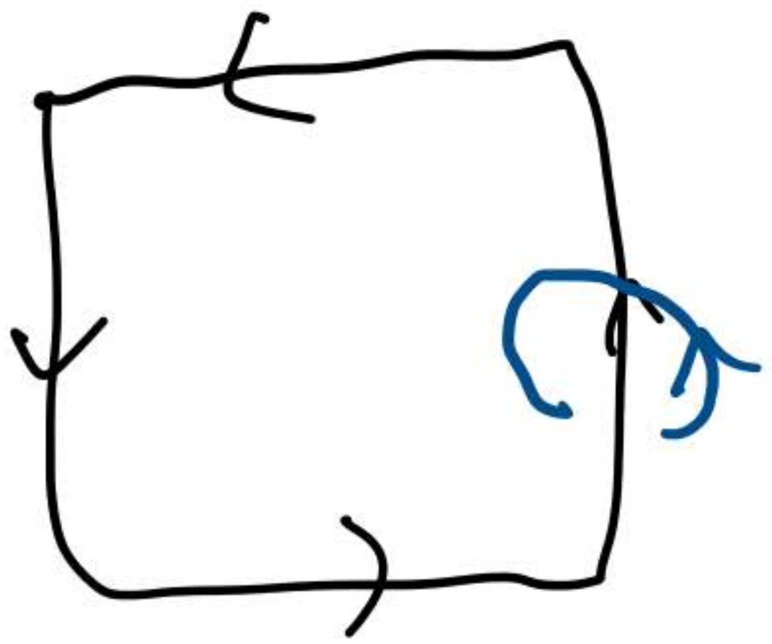
Perturbatively, we get 3d $U(1)^{N-1}$ gauge theory for $a^i = a^{ii}$.

$$\Updownarrow$$

3d compact boson $\vec{\sigma} \sim \vec{\sigma} + 2\pi \vec{\nu}_k$.

$$\mathcal{L} = \frac{g^2}{16\pi^2 L_4} |\partial_\mu \vec{\sigma}|^2 .$$

Wilson loop



$$e^{i \vec{\nu}_k \cdot \oint_C \vec{a}}$$

$$\oint_{S^1} d\vec{\sigma} = 2\pi \vec{\nu}_k$$

Monopole

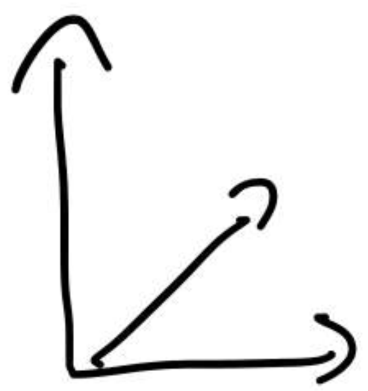
$$\int_{S^2} d\vec{a} = 2\pi \vec{\alpha}_k$$

$$e^{i \vec{\alpha}_k \cdot \vec{\sigma}}$$

Dynamical monopoles

For 3d $SU(N)$ YM + Adj Higgs $\langle \Phi \rangle = \begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_N \end{pmatrix}$ w/ $v_1 > v_2 > \cdots > v_N$.

there are $N-1$ fundamental monopoles



$v_i = v_{i+1}$

$\downarrow$

SU(2) is recovered

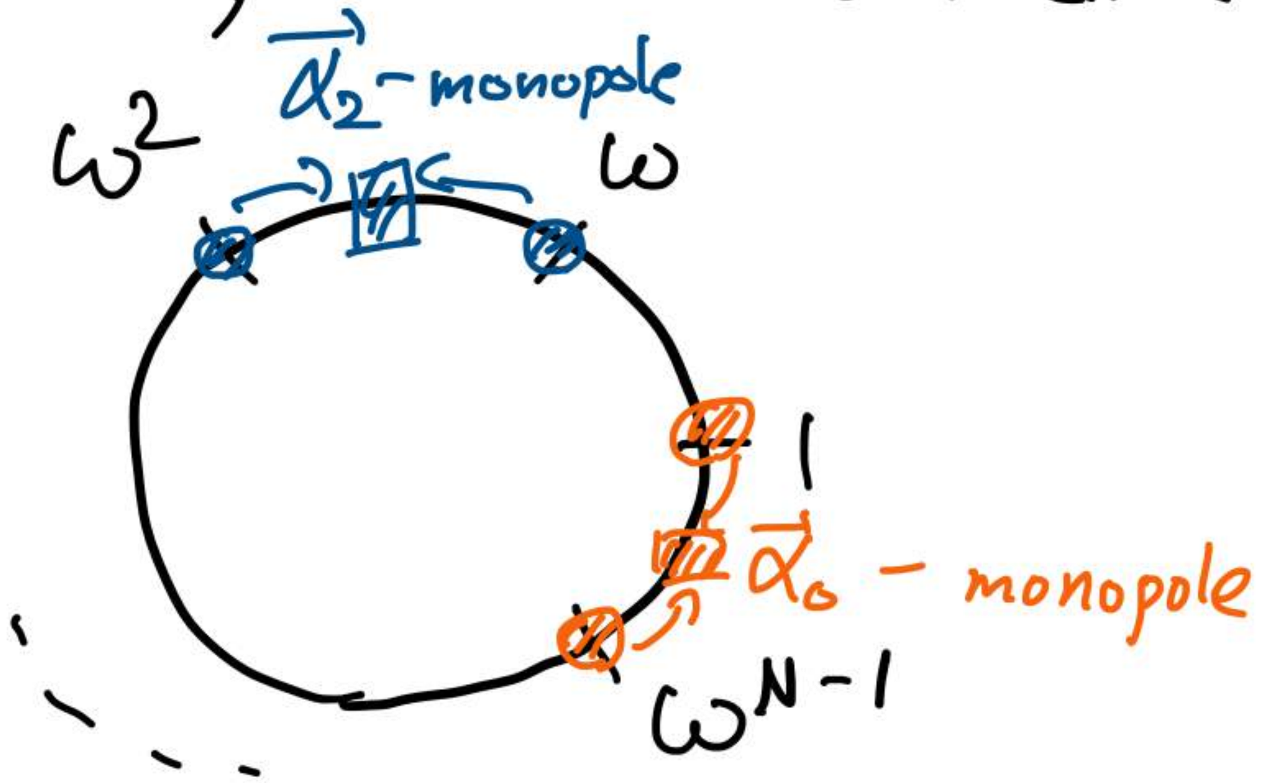
$\Updownarrow$

$\vec{\alpha}_i$ – monopole ('t Hooft-Polyakov monopole)

$v_i \neq v_j$

$SU(N) \to U(1)^{N-1}$

For 4d YM on $\mathbb{R}^3 \times S^1$, there is an extra fundamental monopole (KK monopole):



(Lee, Yi '97
Lee, Lu '98
Kraan, van Baal '98)

3d monopole effective theory

By dilute gas approximation, we obtain

$$Z = \int \mathcal{D}\vec{\sigma} \ \exp\left[-\int d^3x \left\{ \frac{g^2}{L_4} (\partial_\mu \vec{\sigma})^2 - \sum_{n=1}^{N} \underline{\underline{K e^{-\frac{8\pi^2}{g^2 N}}}} \cos\left(\vec{\alpha}_n \cdot \vec{\sigma} + \frac{\theta}{N}\right) \right\}\right]$$

$\Uparrow$

all N monopoles have the same fugacity

because of the $\mathbb{Z}_N^{(0)}$ symmetry

$\vec{\sigma} \to P_W \vec{\sigma}$ (cyclic Weyl permutation)

The monopole potential has N local minima:

$$\vec{\sigma}_* = \frac{2\pi k}{N} \vec{\rho} \quad \left(= \frac{2\pi k}{N} \left((N-1)\vec{\nu}_1 + (N-2)\vec{\nu}_2 + \cdots + \vec{\nu}_{N-1} \right)\right).$$

$\Rightarrow$ N branch structure of confining vacua.

Unification of monopole & center vortex mechanism

(Hayashi, Tanizaki 2024)

Relation between two semiclassical descriptions

4d YM

$\mathbb{R}^2 \times T^2$ w/ 't Hooft flux

& $N L_{3,4} \Lambda \ll 1$

- Center vortex gas

  explains confinement

$$Z = \sum_{k=0}^{N} e^{-\mathrm{Vol}\left[-2K_v e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\frac{\theta - 2\pi k}{N}\right)\right]}$$

$\mathbb{R}^3 \times S^1$ w/ double-trace potential

& $N L_4 \Lambda \ll 1$

- Monopole gas

  explains confinement.

$$Z = \int \mathcal{D}\vec{\sigma}\, e^{-\int \left(\frac{g^2}{L_4}(\partial\vec{\sigma})^2 - \sum_{n=1}^{N-1} 2K_m e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\vec{\alpha}_n \cdot \vec{\sigma} + \frac{\theta}{N}\right)\right)}$$

$\boxed{\text{Can we make a connection between these two?}}$

Let's consider 4d YM in the following setup

- $\mathbb{R}^2 \times \underbrace{S^1_{L_3} \times S^1_{L_4}}_{\text{'t Hooft flux}}$ w/ $\begin{cases} N L_4 \Lambda \ll 1 \\ L_3 \gg L_4 \end{cases}$

- We add the double-trace potential only along $S^1_{L_4}$:

$$\sum \mathrm{tr}(P_4^n)$$

so that $\langle P_4 \rangle = C = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix}$.

$$\Rightarrow \quad \mathcal{L}_{\text{eff}} = \frac{g^2}{L_4}(\partial_\mu \vec{\sigma})^2 - \sum_n 2K_m e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\vec{\alpha}_m \cdot \vec{\sigma} + \frac{\theta}{N}\right).$$

w/ the $\mathbb{Z}_N^{(0)}$-twisted b.c.

$$\vec{\sigma}(\vec{x}, x_3 + L_3) = P_W \cdot \vec{\sigma}(\vec{x}, x_3) \quad \text{mod } 2\pi \vec{\nu}_k.$$

The partition function in this setup is given by

$$Z = \int \mathcal{D}\vec{\sigma} \;\exp\left[-\int d^3x \left\{ \frac{g^2}{L_4}(\partial_\mu \vec{\sigma})^2 - \sum_{n=1}^{N} 2K_m e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\vec{\alpha}_n\cdot\vec{\sigma} + \frac{\theta}{N}\right)\right\}\right]$$

w/ b.c. $\vec{\sigma}(\vec{x}, x_3+L_3) = P_W \cdot \vec{\sigma}(\vec{x}, x_3)$ mod $2\pi \vec{\nu}_k$.

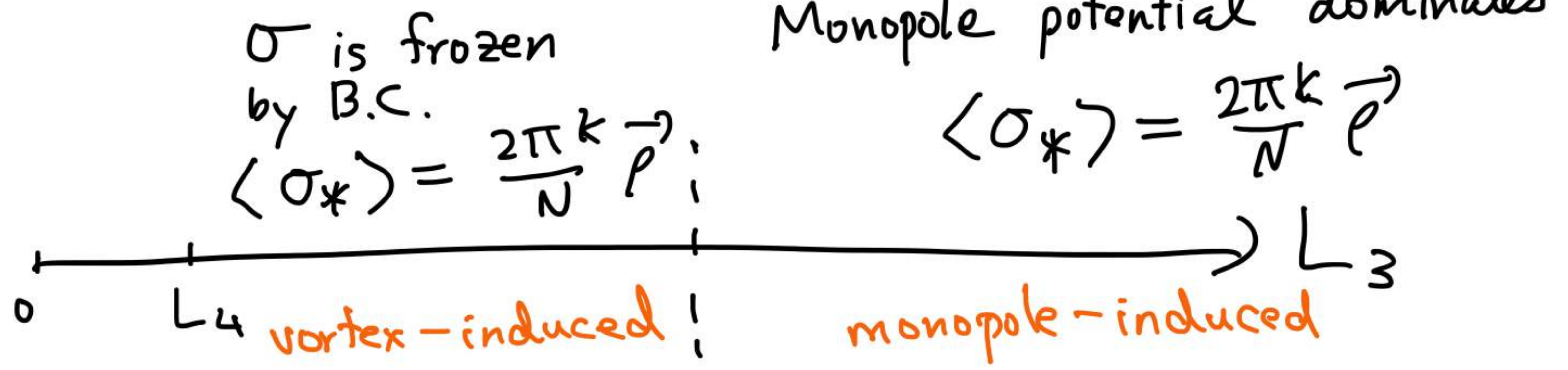
Zero mode of b.c. $\vec{\sigma}(\vec{x}) = P_W \vec{\sigma}(\vec{x})$ mod $2\pi \vec{\nu}_k$

$$\Longleftrightarrow \quad \underline{\vec{\sigma}_* = \frac{2\pi k}{N}\vec{\rho}}$$

This is identical to the local minima of the monopole potential

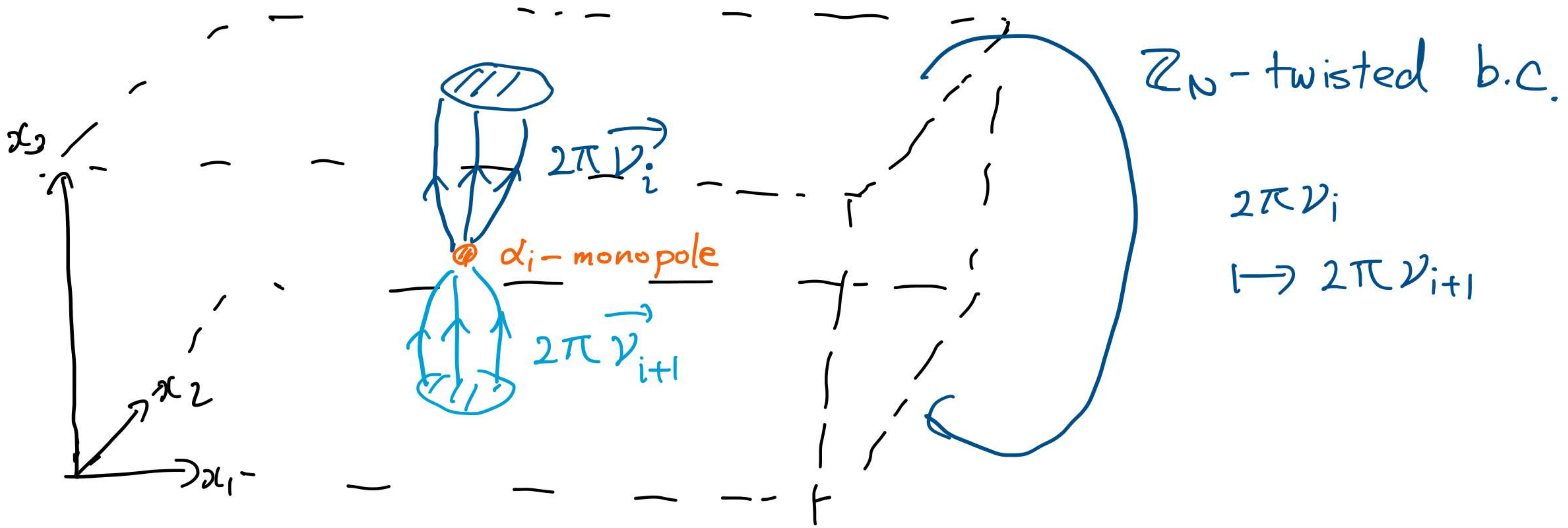
$$\Downarrow$$

No phase transition for L_3.



Microscopic relation between monopole & center vortex

α_i - monopole emits the magnetic flux $2\pi\alpha_i = 2\pi(\nu_i - \nu_{i+1})$.



$\mathbb{Z}_N$-twisted b.c. gives the perturbative gap $\frac{2\pi}{NL_3}$ $\Rightarrow$ Magnetic flux localizes.

$$\boxed{\text{Monopole} = \text{Junction of the center vortex}}$$

(cf. Ambjorn, Giedt, Greensite '99, de Forcrand, Pepe '00)

Summary

(I) Semiclassical approach is developing to understand confinement.

- $\mathbb{R}^3 \times S^1$ w/ double-trace deformation (Ünsal '07, —)

  Monopole is the key player

- $\mathbb{R}^2 \times T^2$ w/ 't Hooft twist (Tanizaki, Ünsal '22, —)

  Center vortex is the key player

(II) Relation between these two descriptions are uncovered (Hayashi, Tanizaki '24)

- Monopole lives at the junction of the center-vortex network