Phases and Duality in Fundamental Kazakov-Migdal Model on the Graph

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Collaboration with
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@ Meiji Gakjuin Univ. Mathematical Informatics NEW!)

Plan of the talk

1. Basics of the graph theory and the graph zeta functions
2. The FKM model and the Ihara zeta functions
3. Duality of the FKM model
4. Instability and the critical strip of the Ihara zeta function
5. GWW phase transitions in the FKM model
6. Numerical results
7. Conclusion and Future works

Basics of the graph theory and the graph zeta functions

Graph and basic concepts

An object that vertices are connected by edges

Set of vertices

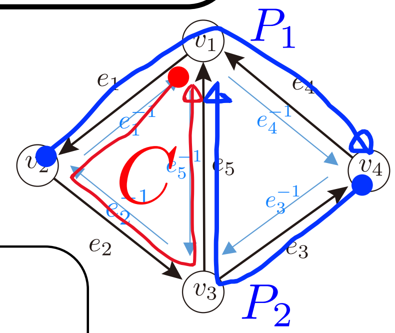
$$V = \{1,2,\cdots,n_V\}$$

Set of edges

$$E = \left\{e_1, e_2, \cdots, e_{n_E}\right\}$$

Source and Target

$$e = \langle v, v' \rangle \in E \Longrightarrow v = s(e), v' = t(e)$$



Inverse edge

$$e = \langle v, v' \rangle \Longrightarrow e^{-1} = \langle v', v \rangle$$

Set of the edges and the inverse edges

$$E_D = \left\{e_1, e_2, \cdots, e_{n_E}, e_1^{-1}, e_2^{-1}, \cdots, e_{n_E}^{-1}\right\} = \{\boldsymbol{e}_a \mid a = 1, \cdots, 2n_E\}$$

Path

$$P = \left(\boldsymbol{e}_{a_1}\boldsymbol{e}_{a_2}\boldsymbol{e}_{a_3}\cdots\boldsymbol{e}_{a_l}\right) : t\left(\boldsymbol{e}_{a_k}\right) = s\left(\boldsymbol{e}_{a_{k+1}}\right)$$

Product of paths

$$P_1 = \left(\boldsymbol{e}_{a_1}\cdots\boldsymbol{e}_{a_l}\right) P_2 = \left(\boldsymbol{e}_{b_1}\cdots\boldsymbol{e}_{b_{l'}}\right) \; t\left(\boldsymbol{e}_{a_l}\right) = s\left(\boldsymbol{e}_{b_1}\right) \Rightarrow P_1P_2 = \left(\boldsymbol{e}_{a_1}\cdots\boldsymbol{e}_{a_l}\boldsymbol{e}_{b_1}\cdots\boldsymbol{e}_{b_{l'}}\right)$$

Cycle

$$C = \left(\boldsymbol{e}_{a_1}\boldsymbol{e}_{a_2}\boldsymbol{e}_{a_3}\cdots\boldsymbol{e}_{a_l}\right), t\left(\boldsymbol{e}_{a_l}\right) = s\left(\boldsymbol{e}_{a_1}\right)$$

Equivalence class of cycles

$$[C] = \left\{\left(\boldsymbol{e}_{a_1}\cdots\boldsymbol{e}_{a_l}\right), \left(\boldsymbol{e}_{a_2}\cdots\boldsymbol{e}_{a_1}\right), \ldots, \left(\boldsymbol{e}_{a_l}\cdots\boldsymbol{e}_{a_1}\right)\right\}$$

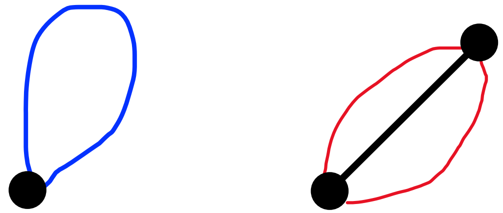
Classification of the graphs

Connected graph

any pair of the vertices can be connected by a path

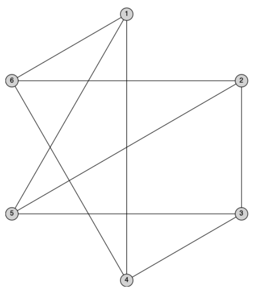
Simple graph

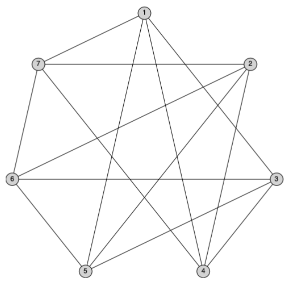
a graph containing no loops nor multiple edges



Regular graph

the degrees of all vertices are the same

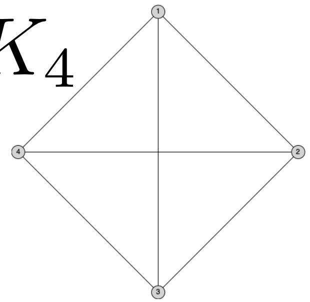




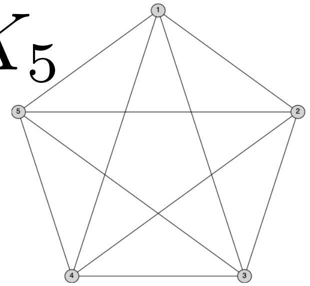
Complete graph

each vertex connects to all other vertices

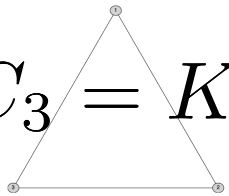
K_4



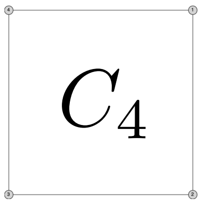
K_5



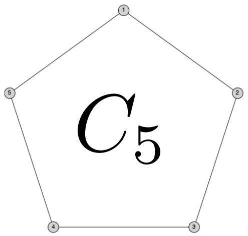
Cycle graph



$C_3 = K_3$



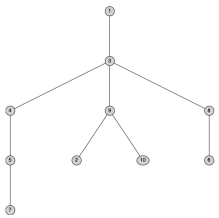
C_4

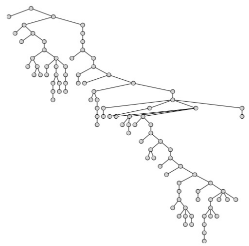


C_5

Tree graph

a graph containing no cycles





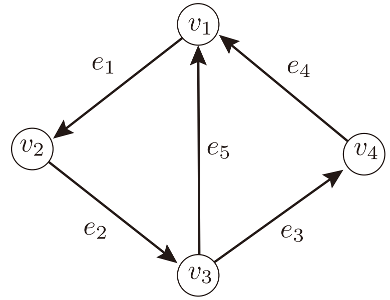
Matrices associated with a graph

Adjacency matrix

$$A_{vv'} = \sum_{\boldsymbol{e} \in E_D} \delta_{\langle v, v' \rangle, \boldsymbol{e}} \cdot \quad (v, v' \in V)$$

$$A_{\mathrm{DT}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(example)
Double Triangle (DT)



Degree matrix

$$D \equiv \mathrm{diag}_{v \in V}(\deg(v)) \quad Q \equiv D - 1$$

$$D_{\mathrm{DT}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Edge adjacency matrix

$$W_{\mathbf{e}\mathbf{e}'} \equiv \begin{cases} 1 & \text{if } t(\mathbf{e}) = s(\mathbf{e}') \text{ and } \mathbf{e}'^{-1} \neq \mathbf{e} \\ 0 & \text{others} \end{cases} . \quad (\mathbf{e}, \mathbf{e}' \in E_D)$$

$W_{\mathrm{DT}} =$

	e_1	e_2	e_3	e_4	e_5	e_1^{-1}	e_2^{-1}	e_3^{-1}	e_4^{-1}	e_5^{-1}
e_1	0	1	0	0	0	0	0	0	0	0
e_2	0	0	1	0	1	0	0	0	0	0
e_3	0	0	0	1	0	0	0	0	0	0
e_4	1	0	0	0	0	0	0	0	0	1
e_5	1	0	0	0	0	0	0	0	1	0
e_1^{-1}	0	1	0	0	0	0	0	0	1	1
e_2^{-1}	0	0	0	0	0	1	0	0	0	0
e_3^{-1}	0	0	0	0	1	0	1	0	0	0
e_4^{-1}	0	0	0	0	0	0	0	1	0	0
e_5^{-1}	0	0	1	0	0	0	1	0	0	0

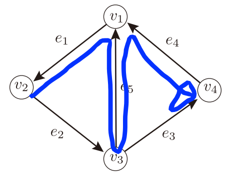
Incidence matrix and Graph Laplacian

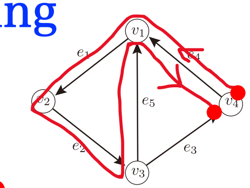
$$L_{ev} \equiv \begin{cases} 1 & v = t(e) \\ -1 & v = s(e) \\ 0 & \text{otherwise} \end{cases} \qquad \Delta \equiv L^T L = D - A$$

$L : V \to E \Leftrightarrow d$: exterior derivative

$$L_{\mathrm{DT}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \qquad \Delta_{\mathrm{DT}} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Cycles





Bumps (backtracking and tail)

In a path $P = (\boldsymbol{e}_{a_1} \cdots \boldsymbol{e}_{a_l})$, a part $(\boldsymbol{e}_{a_i}\boldsymbol{e}_{a_{i+1}})$ satisfying $\boldsymbol{e}_{a_{i+1}} = \boldsymbol{e}_{a_i}^{-1}$: a **backtracking**

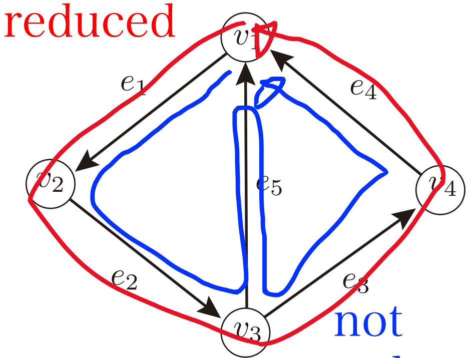
If a cycle $C = (\boldsymbol{e}_{a_1} \cdots \boldsymbol{e}_{a_l})$ satisfies $\boldsymbol{e}_{a_l} = \boldsymbol{e}_{a_1}^{-1}$, this part is called a **tail**

In the equivalence class $[C]$, a backtracking and a tail are the same : a **bump**

Important concepts on cycles

reduced cycle

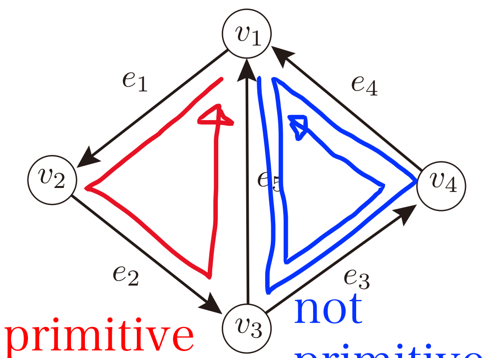
cycle without a bump

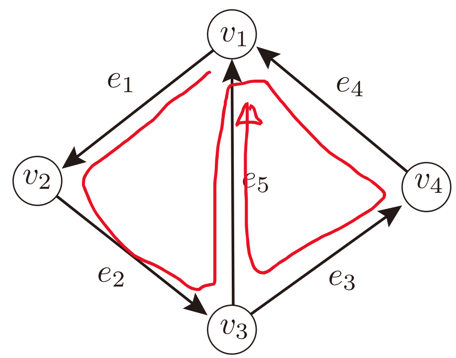


primitive cycle

$$C \neq B^n$$

Note: this is also primitive





The Ihara zeta function

cf) Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

Ihara zeta function

$$\zeta_G(q) \equiv \prod_{[C]:\text{primitive reduced}} \frac{1}{1-q^{|C|}}$$

length of the cycle

Ihara zeta function is expressed as the reciprocal of a finite polynomial

Vertex expression

Ihara zeta function is the inverse of the determinant of a finite matrix (characteristic polynomial): Ihara 1966

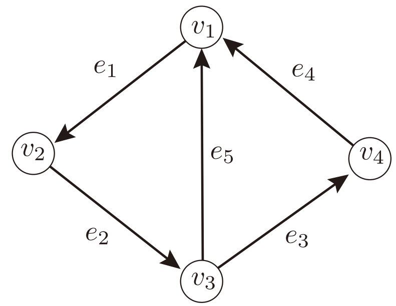
$$\zeta_G(q) = (1-q^2)^{-(n_E - n_V)} \det\left(I - qA + q^2 Q\right)^{-1}$$

Edge expression

Equivalently, Hashimoto 1990, Bass 1992

$$\zeta_G(q) = \det(1 - qW)^{-1}$$

How is it possible?



Ihara zeta function for Double-Triangle

$$\zeta_{\mathrm{DT}}(q)^{-1} = 1 - 4q^3 - 2q^4 + 4q^6 + 4q^7 + q^8 - 4q^{10}$$

Observation

length	3	4	6	7	9	10	11	12	13	14	15
#(PRC)	4	2	2	4	4	12	4	6	32	18	16

$$\prod_{[C],|C|\leq 15} (1 - q^{|C|}) = 1 - 4q^3 - 2q^4 + 4q^6 + 4q^7 + q^8 - 4q^{10} + 80q^{16} + \mathrm{O}(q^{17})$$

Higher order terms disappear by increasing the upper limit of $|C|$ in the product.

Important properties of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(1) Euler product

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

(2) functional equation and the critical strip

Completed zeta function: $\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ ➡ $\xi(1-s) = \xi(s)$

critical strip : $0 < s < 1$

(3) Riemann's hypothesis

Non-trivial zeros of $\zeta(s)$ are only on $Re(s) = \frac{1}{2}$

Is Ihara zeta function a zeta function?

(1) Euler product?

$$\zeta_G(q) \equiv \prod_{[C]:PR} \frac{1}{1-q^{|C|}}$$

(2) functional equation?

- If the graph is (t+1)-regular, $\zeta_G(q) = (1-q^2)^{-(n_E-n_V)} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)^{-1}$
- Completed Ihara zeta： $\xi_G(q) \equiv (1-q^2)^{n_E-\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}}\zeta_G(q) = (1-q^2)^{\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)$

$$\xi_G\left(\frac{1}{tq}\right) = (-1)^{n_V}\xi_G(q)$$

(3) Riemann's hypothesis?

If the graph is Ramanujan, non-trivial zeros of $\zeta_G(t^{-s})$ are only on $Re(s) = \frac{1}{2}$

(Ramanujan graph：(t+1)-regular and the eigenvalues of A except for $t+1$ satisfies $\lambda^2 < 4t$)

proof

Ihara zeta function of (t+1)-regular graph:

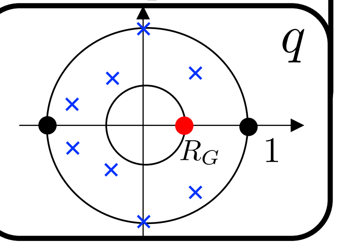
$\zeta_G(q) = (1-q^2)^{n_V-n_E} \det\left((1-tq^2)\mathbf{1}_{n_V} - qA\right) = (1-q^2)^{n_V-n_E}\prod_{\lambda}(tq^2-\lambda q+1)$ ➡ zeros：$q = \frac{\lambda \pm \sqrt{\lambda^2-4t}}{2t} \equiv t^{-s_\pm}$

If $\lambda^2 - 4t < 0$, since $s_- = s_+^*$, $t^{-s_+} \cdot t^{-s_-} = t^{-s_+-s_-} = t^{-2Re(s_+)} = t^{-1}$

Critical strip of the Ihara zeta function

(a part of) Kotani-Sunada's theorem

- $\zeta_G(q)$ has a first pole at $q = R_G < 1$ which has the smallest magnitude of all poles.
- The poles of $\zeta_G(q)$ exist in $R_G \leq |q| \leq 1$
- When G is $(t+1)$-regular, $R_G = 1/t$

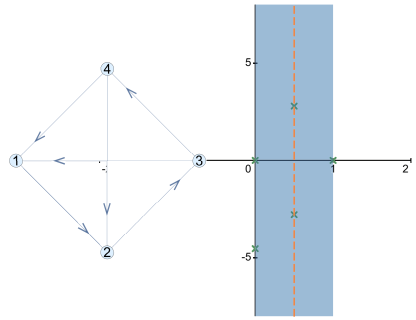


Redefinition of the parameter

$q \equiv R_G^s$ or $s \equiv \frac{\log q}{\log R_G}$ ➡ all poles exist in $0 \leq Re(s) \leq 1$ **(critical strip)**

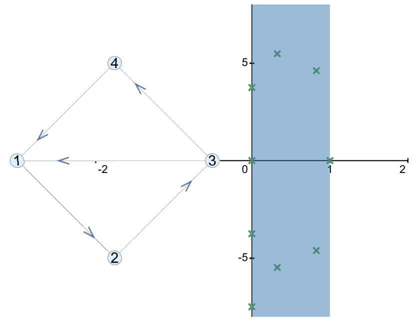
Tetrahedron (K_4)

$R_{K_4} = \frac{1}{t} = \frac{1}{2}$



Double Triangle ($K_4 - e$)

$R_{DT} \simeq 0.65729\cdots$



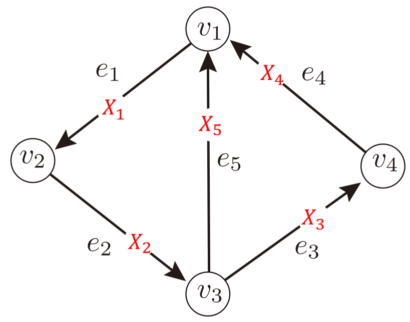
Matrix weighted Ihara zeta function

preparation

Ohta-S.M. 2022

cf) Mizuno, Sato 2003,2006

- regular matrix X_e (size K) on each edge e
- $X_{e^{-1}} = X_e^{-1}$
- $X_C \equiv X_{e_{i_1}} \cdots X_{e_{i_n}}$ for $C = e_{i_1} \cdots e_{i_n}$



Matrix weighted Ihara zeta function Ohta-S.M. 2022

$$\zeta_G(q;X) \equiv \prod_{C \in [\mathcal{P}_R]} \det\left(1_K - q^{|C|} X_C\right)^{-1}$$

matrix weighted adjacency matrices and the matrix weighted Ihara zeta function

$$A(X)_{vv'} = \begin{cases} X_e & \langle v, v' \rangle = e \\ 0 & \text{others} \end{cases} \qquad (W_X)_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} X_e & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e}') \text{ and } \boldsymbol{e}'^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}$$

$$\zeta_G(q;X) = \left(1 - q^2\right)^{-K(n_E - n_V)} \det\left(1_{Kv_N} - qA_X + q^2 Q\right)^{-1}$$: vertex expression

$$= \det\left(1_{2Kn_E} - qW_X\right)^{-1}$$: edge expression

Ihara zeta function as a generating function of reduced cycles

#(reduced cycle of length n)

$$\zeta_G(q) = \exp\left(-\sum_{[C]:\mathrm{PR}} \log(1-q^{|C|})\right) = \exp\left(\sum_{[C]:\mathrm{PR}} \sum_{m=1}^{\infty} \frac{q^{m|C|}}{m}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} q^n\right)$$

The Ihara zeta function counts the number of reduced cycles

Generalization to the unitary matrix weighted Ihara zeta function

$$\zeta_G(q;U) \equiv \prod_{C\in[\mathcal{P}_R]} \det\left(\mathbf{1}_{N_c} - q^{|C|}U_C\right)^{-1} = \exp\left(\sum_{C\in[\Pi_+]} \sum_{n=1}^{\infty} \frac{q^n}{n}\left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger\, n}\right)\right)$$

"chiral" primitive reduced cycles

The U-Ihara zeta function counts all Wilson loops on the graph

The Bartholdi zeta function

Bartholdi 2000

$$\zeta_G(q,u) \equiv \prod_{[C]:\text{primitive}} \frac{1}{1-q^{|C|}u^{b(C)}}$$

#(bumps)

$\left(\zeta_G(q,u=0)=\zeta_G(q)\right)$

Vertex expression

$$\zeta_G(q,u) = \left(1-(1-u)^2q^2\right)^{-(n_E-n_V)} \det\left(1-qA+(1-u)q^2(D-(1-u)1)\right)^{-1},$$

Edge expression

$$\zeta_G(q,u) = \det(1-q(W+uJ))^{-1}$$

$$J_{ee'} = \begin{cases} 1 & (\boldsymbol{e'}^{-1}=\boldsymbol{e}) \\ 0 & (\text{others}) \end{cases}$$

Properties of the Bartholdi zeta function

Ohta-S.M. arXiv:2408.04952 (math.CO)

Functional equations w.r.t q and u $\quad G:\ (t+1)$ −regular graph

$$\begin{cases} \zeta_G(1/(1-u)(t+u)q, u) = (-1)^{n_E-n_V}(1-u)^{n_V}(t+u)^{2n_E-n_V}q^{2n_E}\left(\dfrac{1-(1-u)^2q^2}{1-(t+u)^2q^2}\right)^{n_E-n_V}\zeta_G(q,u) \\ \zeta_G(q, 1-t-u) = \left(\dfrac{1-(1-u)^2q^2}{1-(t+u)^2q^2}\right)^{n_E-n_V}\zeta_G(q,u) \end{cases}$$

(Guido-Isola-Lapidus, 2008)

$$u_* = \frac{1-t}{2}$$

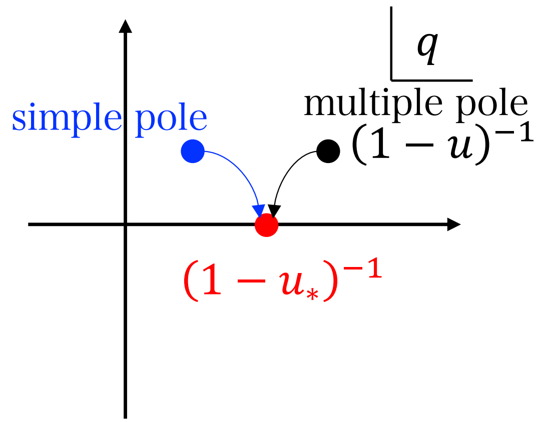
Equivalence between Ihara and Bartholdi $\quad \zeta_G(q) = \left(\dfrac{1-t^2q^2}{1-q^2}\right)^{n_E-n_V}\zeta_G(q,1-t)$

Poles of the Bartholdi zeta function (for a general graph)

Theorem 7. *The order of the pole at $q=(1-u)^{-1}$ of $\zeta_G(q,u)$ is n_E-n_V+1 if $u\neq u_*$, while it is enhanced to more than or equal to n_E-n_V+2 only if $u=u_*$. In particular, if the graph satisfies*

$$|(L^+)^T\vec{d}|^2 \neq n_E\,, \tag{4.51}$$

the order of the pole at $q=(1-u)^{-1}$ enhances exactly to n_E-n_V+2 when $u=u_*$.



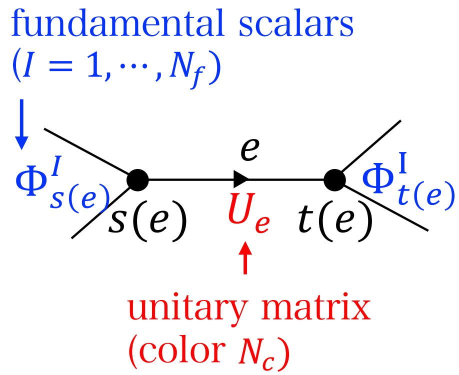
The FKM model
and
the Ihara zeta function

FKM model

Arefeva 1993
Ohta-S.M. 2023

Fundamental Kazakov-Migdal (FKM) model on a general graph

$$S = \sum_{v \in V} m_v^2 \Phi_v^{\dagger I} \Phi_{vI} - q \sum_{e \in E} \left(\Phi_{s(e)}^{\dagger\ I} U_e \Phi_{t(e)I} + \Phi_{t(e)}^{\dagger\ I} U_e^\dagger \Phi_{s(e)I} \right)$$



cf) KM model on the graph

Kazakov-Migdal 1992
Ohta-S.M. 2022

$$S_{\mathrm{KM}} = \mathrm{Tr}\left\{ \frac{m_0^2}{2} \sum_{v \in V} \Phi_v^2 + q \sum_{e \in E} \left(\frac{r}{2} \left(\Phi_{s(e)}^2 + \Phi_{t(e)}^2 \right) - \Phi_{s(e)} U_e \Phi_{t(e)} U_e^\dagger \right) \right\}$$

The partition function after tunning the mass parameters: $m_v^2 = 1 + (\deg v + 1) q^2$

$$Z_G = \mathcal{N} \int \prod_{e \in E} dU_e \, \zeta_G(q; U)^{N_f}$$

$$\left(\mathcal{N} = (2\pi)^{N_f N_c n_V} \left(1 - q^2\right)^{N_f N_c (n_E - n_V)} \right)$$

FKM model is described by the unitary matrix weighted graph zeta function

Effective action and the relation to the Wilson action

$$\zeta_G(q;U) = \exp\left(\sum_{C\in[\Pi_+]}\sum_{n=1}^{\infty}\frac{q^n}{n}\left(\mathrm{Tr}\,U_C^n + \mathrm{Tr}\,U_C^{\dagger\,n}\right)\right)$$

$$S_{\mathrm{eff}}(U) = -N_f\sum_{C\in[\Pi_+]}\sum_{n=1}^{\infty}\frac{q^n}{n}\left(\mathrm{Tr}\,U_C^n + \mathrm{Tr}\,U_C^{\dagger\,n}\right)$$

※ valid only for small $|q|$
(at most $|q| < 1$)

$\gamma \equiv N_f/N_c \quad q \to 0, \quad \gamma \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l}$: fixed (l : minimal length of the cycles)

$$S_{\mathrm{eff}}(U) \to -\frac{N_c}{\lambda}\sum_{C\,:\,\text{minimal length}}\left(\mathrm{Tr}\,U_C + \mathrm{Tr}\,U_C^{\dagger}\right)$$

FKM model is a generalization of the usual lattice gauge theory

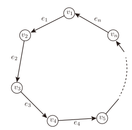
Duality
of the FKM model

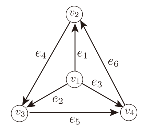
Duality of the FKM model

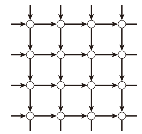
Ohta-S.M. 2024

functional equation of the U-Ihara zeta function for a $(t+1)$-regular graph

$$\zeta_G(1/tq;U) = (tq^2)^{n_V N_c} \left(\frac{-tq^2(1-q^2)}{1-t^2q^2} \right)^{(n_E - n_C)N_c} \zeta_G(q;U)$$







The FKM model on a regular graph is **self-dual**
thanks to the property of the zeta function

"functional relation" for an irregular graph

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$$\left(\tilde{Q} \equiv \mathrm{diag}_e(\deg s(e) - 1)\right)$$

For an arbitrary ω,

$$\zeta_G(1/\omega q;U) \propto \det\left(1 - \omega q \left(\tilde{Q}^{-1} W_U - (1-\tilde{Q}^{-1}) J_U\right)\right)^{-1} = \exp\left(\sum_{C \in [\Pi_+]} \sum_{n=1}^{\infty} {}^{\exists} f_{C,n}(q) \left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger n} \right) \right)$$

a matrix weighted Bartholdi zeta function with (unfamiliar) weights

The FKM model on an irregular graph has also a **dual expression**.

Remarks

1) This duality disappears in the Wilson limit:

FKM → Wilson

$$\gamma \equiv N_f/N_c \quad q \to 0, \quad \gamma \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l} \ : \ \text{fixed}$$

$$S_{\text{eff}}(U) \to -\frac{N_c}{\lambda} \sum_{C \in [\Pi_+^l]} \left(\text{Tr}\, U_C + \text{Tr}\, U_C^\dagger \right)$$

Since the Wilson limit appear in $q \to 0$, the dual description is infinitely separated in this limit.

All Wilson loops seem to be required for the duality to emerge.
(Stringy effect ?)

2) The original Kazakov-Migdal model also has this duality.

As far as I know, this has been missed.

Instability
and
the critical strip
of the Ihara zeta function

Stability of the FKM model

The action of the FKM model

$$S = \sum_{v \in V} m_v^2 \Phi_v^{\dagger I} \Phi_{vI} - q \sum_{e \in E} \left(\Phi_{s(e)}^{\dagger}{}^{I} U_e \Phi_{t(e)I} + \Phi_{t(e)}^{\dagger}{}^{I} U_e^{\dagger} \Phi_{s(e)I} \right) \equiv \Phi_v^{I\dagger} \Delta(q;U)_{vv'} \Phi_{v'}^{I}$$

$$\Delta(q;U) \equiv 1 - qA_U + q^2 Q$$

The kernel at the vacuum $U = 1$

$$\Delta(q;U) \underset{U \to 1}{\to} 1 - qA + q^2 Q \equiv \Delta(q)$$: deformed graph Laplacian

- $U = 1$ is stable when $\Delta(q)$ is positive definite
- $\det \Delta(q) = \prod_{i=1}^{n_V} \lambda_i(q) = (1-q^2)^{-n_E+n_V} \zeta_G(q)^{-1}$

fact 1: $\Delta(q) \underset{q \to 0}{\longrightarrow} 1$ and $\Delta(q) \underset{|q| \to \infty}{\longrightarrow} q^2 Q = q^2 \,\mathrm{diag}_{v \in V}(\deg v - 1)$

fact 2: $\zeta_G(q)$ has poles at $q = 1$ and R_G and the other poles are in $R_G \leq |q| \leq 1$ (critical strip)

For $q > 0$, the FKM model is stable in $q < R_G$ and $q > 1$

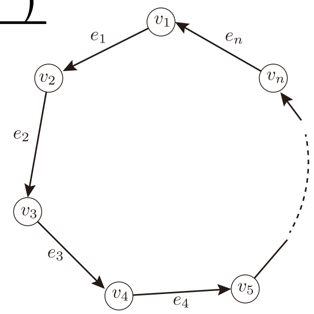
GWW phase transition in the FKM model

GWW phase transition on the cycle graph

After gauge fixing : $U_2 = \cdots = U_n = 1 \ (U_1 \equiv U, \alpha \equiv q^n)$

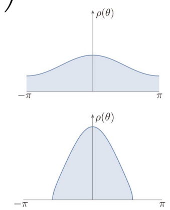
$$Z_{C_n} = \mathcal{N} \int dU \, e^{N_f \sum_{m=1}^{\infty} \frac{\alpha^m}{m} \left(\mathrm{Tr}\, U^m + \mathrm{Tr}\, U^{-m}\right)}$$

a one-matrix model solvable in large N



Eigenvalue density in large N_c $\left(\rho(\theta) \equiv \frac{1}{N_c} \sum_{i=1}^{N_c} \delta(\theta - \theta_i)\right)$

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi}\left(1 + 2\gamma \frac{\alpha\cos\theta - \alpha^2}{1 - 2\alpha\cos\theta + \alpha^2}\right), & (\theta_0 = \pi) \\ \frac{2(\gamma-1)\alpha}{\pi} \frac{\cos\frac{\theta}{2}}{1 - 2\alpha\cos\theta + \alpha^2} \sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}, & (\theta_0 < \pi) \end{cases}$$



$\left(\sin^2\frac{\theta_0}{2} = \frac{(1-\alpha)^2}{4\alpha}\frac{2\gamma-1}{(\gamma-1)^2}\right)$

Free energy

$$F_{C_n} \equiv -\lim_{N_c \to \infty} \frac{1}{N_c^2} \log Z_{C_n}$$

$$= \begin{cases} F_{C_n}^- \equiv \gamma^2 \log\left(1 - \alpha^2\right) & (0 < \alpha \le \alpha^*) \\ F_{C_n}^+ \equiv (2\gamma - 1)\log(1-\alpha) + \frac{1}{2}\log\alpha + f(\gamma) & (\alpha^* < \alpha \le 1) \end{cases} \quad \left(\alpha^* = \frac{1}{2\gamma - 1}\right)$$

Wilson limit

$q \to 0, \quad \gamma \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l}$: fixed

$$\rho(\theta) \to \begin{cases} \frac{1}{2\pi}\left(1 + \frac{2}{\lambda}\cos\theta\right), & (\theta_0 = \pi) \\ \frac{2}{\pi\lambda}\cos\frac{\theta}{2}\sqrt{\frac{\lambda}{2} - \sin^2\frac{\theta}{2}}, & (\theta_0 < \pi) \end{cases}$$

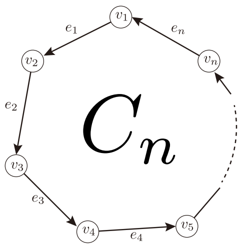
$$F_{C_n} \to \begin{cases} F_{C_n}^- \equiv -\frac{1}{\lambda^2} & (0 < \alpha \le \alpha^*) \\ F_{C_n}^+ \equiv -\frac{2}{\lambda} - \frac{1}{2}\log\frac{\lambda}{2} + \frac{3}{4} & (\alpha^* < \alpha < 1) \end{cases},$$

exactly reproduces
the result of GWW model

3rd order GWW phase transition (appears only for $\gamma > 1$)

(valid only in $q < 1$)

Phase in the dual side



C_n : regular graph ➡ Ihara zeta function is self-dual ($t = 1$)

$$\zeta_{C_n}(1/q;U) = q^{-2nN_c}\zeta_{C_n}(q;U)$$

Free energy in $q > 1$

$$F_{C_n}(1/q) = F_{C_n}(q) - 2n\gamma \log q$$

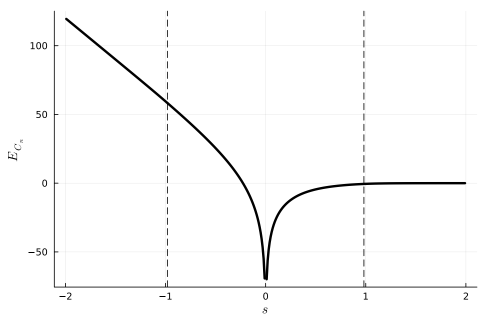
parametrization for C_n

$$s = -\log q$$

(cf) for a $(t+1)$-regular graph $s \equiv \frac{\log q}{\log R_G} = -\frac{\log q}{\log t}$)

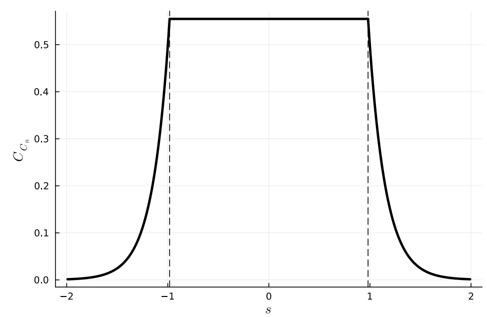
internal energy

$$E_G \equiv \gamma \frac{\partial F_G}{\partial \gamma}$$



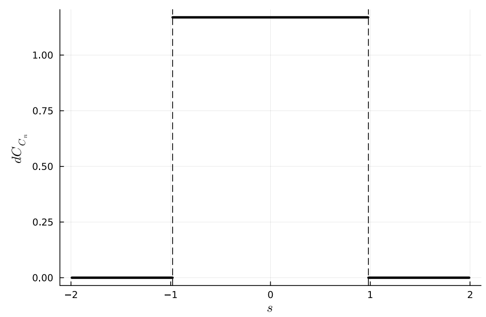
specific heat

$$C_G \equiv -\gamma^2 \frac{\partial^2 F_G}{\partial \gamma^2}$$



derivative of the specific heat

$$dC_G \equiv \gamma^3 \frac{\partial^3 F_G}{\partial \gamma^3}$$



※ $n = 3, \gamma = 10 \Rightarrow s_* \simeq \pm 0.98$

Phase structure in the general graph

<u>effective action in general</u>

$$S_{\text{eff}}(U) = -N_f \sum_{C \in [\Pi_+]} \sum_{n=1}^{\infty} \frac{q^n}{n} \left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger n} \right)$$

: multi-matrix model

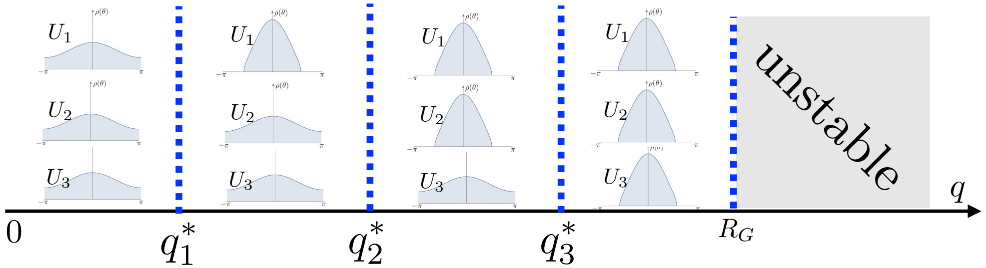
hard to solve even in large N in general

How to analyze?

Hint 1 : all cycles are generated by fundamental cycles and the GWW phase transitions will take place for each fundamental cycle at some coupling constant.

<u>example</u>

3 fund. cycles



Hint 2 : For C_n, the position of the phase transition can be almost read off in the Wilson limit

$$\gamma \to \infty, \quad q \to 0, \quad \gamma q^n \equiv \lambda^{-1} \ : \text{fixed} \quad \Rightarrow \quad q^* \simeq (2\gamma)^{-1/n}$$

<u>Strategy</u>

Scale the parameters so that the action can be regarded as the GWW model (if it possible).

Fundamental cycles and the GWW phase transition

fundamental cycles of the minimal lengths $(r = n_E - n_V + 1)$

$$\mathcal{F} \equiv \{[C_a]\,|\,a = 1, \cdots, r,\ |C_1| \le \cdots \le |C_r|\}$$

set of l_i-gons in $\mathcal{F}$

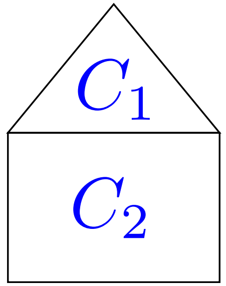
$$\mathcal{F}_{l_i} \equiv \{[C_a] \in \mathcal{F}\,|\,|C_a| = l_i\}, \quad m_{l_i} \equiv |\mathcal{F}_{l_i}|$$

a subset of $[\Pi_+]$ whose length is l_i but are not in $\mathcal{F}_{l_i}$

$$\mathcal{G}_{l_i} \equiv \{[C] \in [\mathcal{P}_R]\,|\,|C| = l_i,\ [C] \notin \mathcal{F}_{l_i}\}$$

Triangle-Square(TS)

$r = 2$



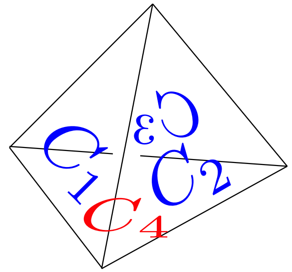
$\mathcal{F}_3 = \{[C_1]\}$

$\mathcal{F}_4 = \{[C_2]\}$

$\mathcal{G}_3 = \mathcal{G}_4 = \{\emptyset\}$

Tetrahedron (K4)

$r = 3$



$\mathcal{F}_3 = \{[C_1], [C_2], [C_3]\}$

$\mathcal{G}_3 = \{[C_4]\} = \{[C_1 C_2 C_3]\}$

the effective action of the FKM model in the region $0 < q < R_G$

$$S_{\text{eff}}(q;U) = -\gamma N_c \,\mathrm{Tr}\left[\sum_{i=1}^{b_G} q^{l_i}\left(\sum_{C_a \in \mathcal{F}_{l_i}} (U_{C_a} + U_{C_a}^{-1}) + \sum_{C' \in \mathcal{G}_{l_i}} (U_{C'} + U_{C'}^{-1})\right) + \mathcal{O}(q^{l_{b_G}+1})\right]$$

$$S_{\text{eff}}(q;U) = -\gamma N_c \,\mathrm{Tr}\Big(q^3 (U_{C_1} + U_{C_1}^{-1}) + q^4 (U_{C_2} + U_{C_2}^{-1}) + \cdots\Big)$$

$$S_{\text{eff}}(q;U) = -\gamma N_c \,\mathrm{Tr}\Big(q^3 (U_{C_1} + U_{C_2} + U_{C_3} + U_{C_1} U_{C_2} U_{C_3} + h.c.) + \cdots\Big)$$

In large γ :

$\mathcal{G}_{l_i} = \emptyset$ ➡ GWW transition around $q_i^* \simeq (2\gamma)^{1/l_i}$ is expected

$\mathcal{G}_{l_i} \neq \emptyset$ ➡ We cannot use the result of GWW model. **Numerical simulation necessary**

Phase in the dual side

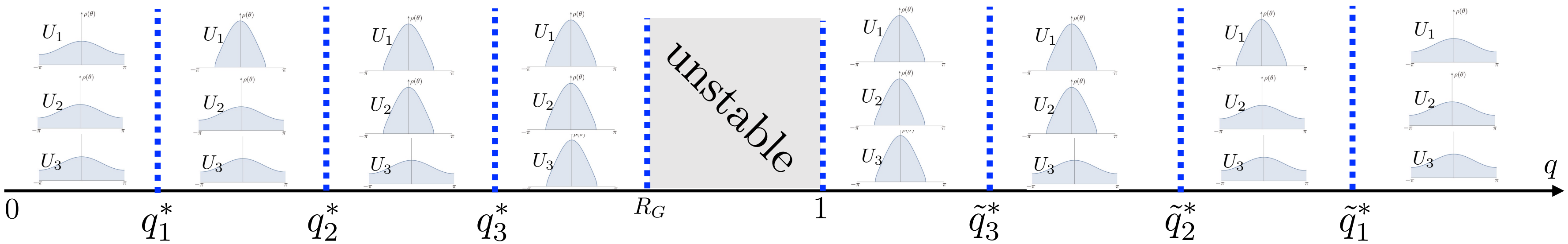
recall

$$S_{\text{eff}}(U) = -N_f \sum_{C\in[\Pi_+]} \sum_{n=1}^{\infty} \frac{q^n}{n}\left(\operatorname{Tr} U_C^n + \operatorname{Tr} U_C^{\dagger\, n}\right)$$: valid only for small q

graph is regular ➡ $q > 1$ is just a copy of $0 < q < R_G$ because of the exact duality

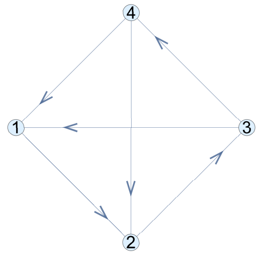
graph is irregular ➡ we have to write down the $1/q$-expansion of the dual action through $\zeta_G(R_G/q;U) \propto \det\left(1 - \frac{q}{R_G}\left(\tilde{Q}^{-1}W_U - (1-\tilde{Q}^{-1})J_U\right)\right)^{-1}$

$$\tilde{S}_{\text{eff}}(q;U) = -\gamma N_c \operatorname{Tr}\left[\sum_{i=1}^{b_G} g_i(q)\left(\sum_{C_a\in\mathcal{F}_i}(U_{C_a}+U_{C_a}^{-1}) + \sum_{C'\in\mathcal{G}_i}(U_{C'}+U_{C'}^{-1})\right) + \mathcal{O}(q^{-l_{b_G}-1})\right]$$ (for $q > 1$)



Examples

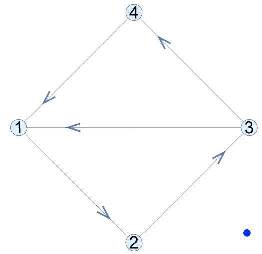
(1) Tetrahedron (3-regular)



$$S_{\text{eff}}(q;U) = \tilde{S}_{\text{eff}}(3/q;U)$$
$$= -\gamma N_c \operatorname{Tr}\left[q^3 \left(U_1 + U_2 + U_3 + U_1 U_2 U_3 + \text{h.c.}\right) + \mathcal{O}(q^{-4})\right]$$

we will see the exact duality but will be different from the GWW model

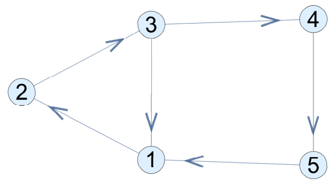
(2) Double Triangle (irregular)



$$S_{\text{eff}}(q;U) = -\gamma N_c \operatorname{Tr}\left[\sum_{a=1,2} q^3 \left(U_a + U_a^{-1}\right) + \mathcal{O}(q^{-4})\right]$$
$$\tilde{S}_{\text{eff}}(q;U) = -\gamma N_c \operatorname{Tr}\left[g(q) \sum_{a=1,2} \left(U_a + U_a^{-1}\right) + \mathcal{O}(q^{-4})\right]$$
$$g(q) = \frac{1}{4q^3} - \frac{3}{16q^5} + \frac{17}{64q^7} - \frac{63}{256q^9} + \mathcal{O}(q^{-11})$$

- one transition in the both region
- both regions will be similar

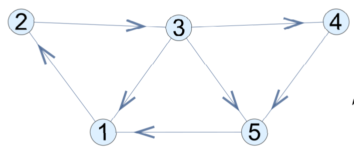
(3) Triangle-Square (irregular)



$$S_{\text{eff}}(q;U) = -\gamma N_c \operatorname{Tr}\Big(q^3 \left(U_{C_1} + U_{C_1}^{-1}\right) + q^4 \left(U_{C_2} + U_{C_2}^{-1}\right) + \cdots\Big)$$
$$\tilde{S}_{\text{eff}}(q;U) = -\gamma N_c \operatorname{Tr}\left[\sum_{a=1,2} g_a(q) \left(U_a + U_a^{-1}\right) + \mathcal{O}(q^{-5})\right]$$
$$g_1(q) = \frac{1}{4q^3} - \frac{3}{16q^5} + \frac{17}{64q^7} - \frac{39}{256q^9} + \mathcal{O}(q^{-11}),$$
$$g_2(q) = \frac{1}{4q^4} - \frac{3}{16q^6} + \frac{17}{64q^8} - \frac{63}{256q^{10}} + \mathcal{O}(q^{-12}).$$

- two transitions in both sides
- both regions will be similar

(4) Triple Triangle (irregular)

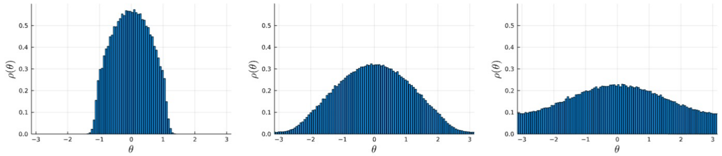


$$S_{\text{eff}}(q;U) = -\gamma N_c \operatorname{Tr}\left[\sum_{a=1,2,3} q^3 \left(U_a + U_a^{-1}\right) + \mathcal{O}(q^{-4})\right]$$
$$\tilde{S}_{\text{eff}}(q;U) = -\gamma N_c \operatorname{Tr}\Big[g_1(q) \left(U_1 + U_2 + U_1^{-1} + U_2^{-1}\right) + g_2(q) \left(U_3 + U_3^{-1}\right) + \mathcal{O}(q^{-4})\Big]$$
$$g_1(q) = \frac{1}{6q^3} - \frac{11}{72q^5} + \frac{233}{864q^7} + \mathcal{O}(q^{-9}),$$
$$g_2(q) = \frac{1}{12q^3} - \frac{17}{144q^5} + \frac{115}{576q^7} + \mathcal{O}(q^{-9}).$$

- one transition in $q < 1$
- two transitions in $q > 1$

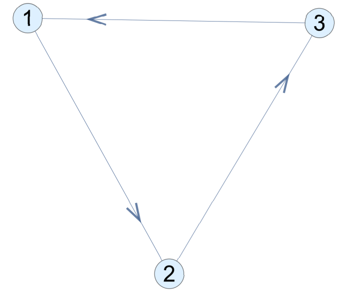
Numerical results

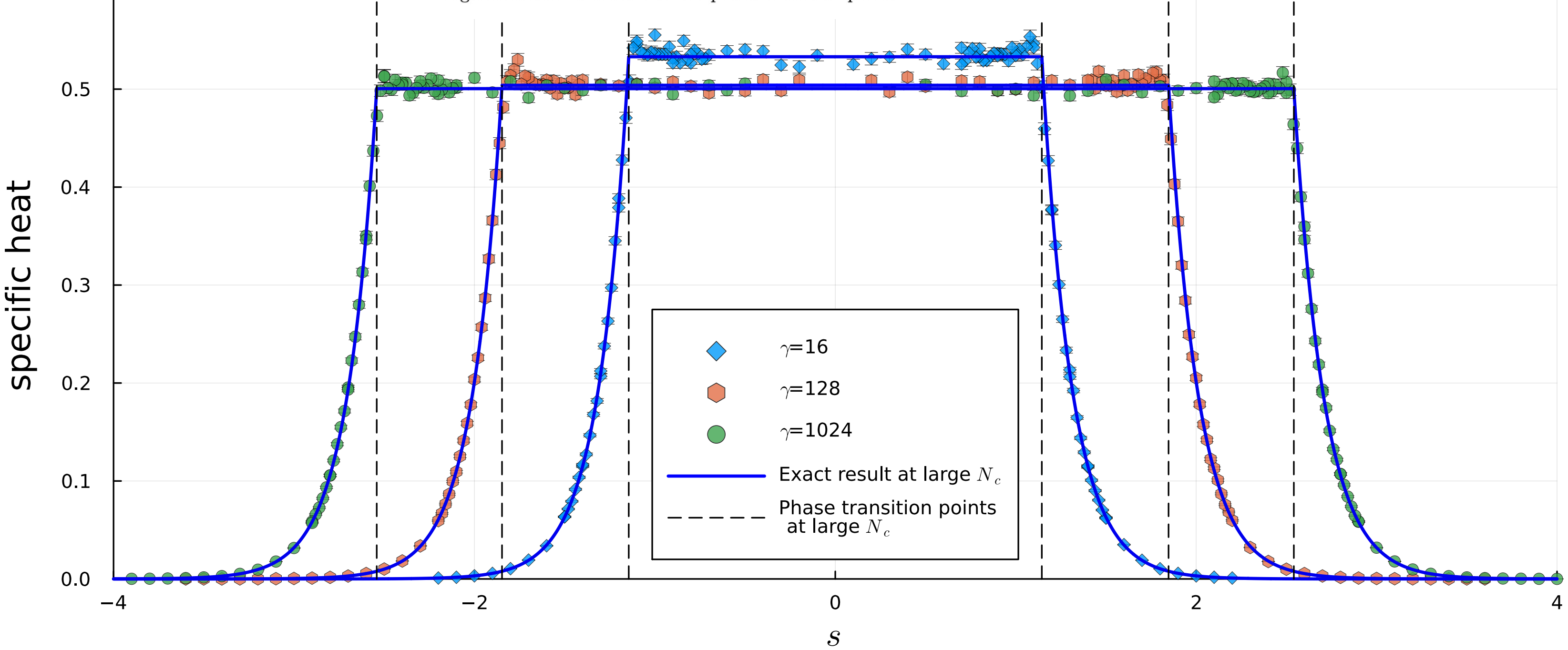
(0) Cycle graph (C_3)



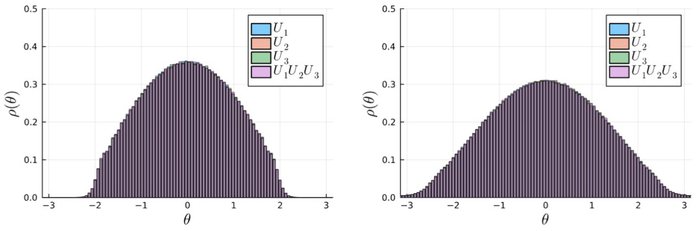
(a) $s = 2.15848$ (b) $s = 2.54084 \simeq s^*$ (c) $s = 2.84215$

Figure 10: The eigenvalue distributions of U for $N_c = 16$ and $\gamma = 1024$. (a) and (c) corresponds to the deconfinement and confinement phase, respectively. (b) is the eigenvalue distribution near the phase transition point.



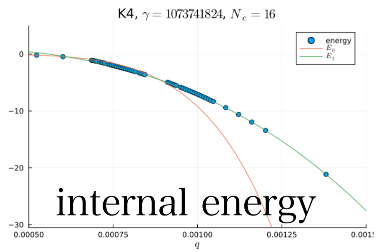


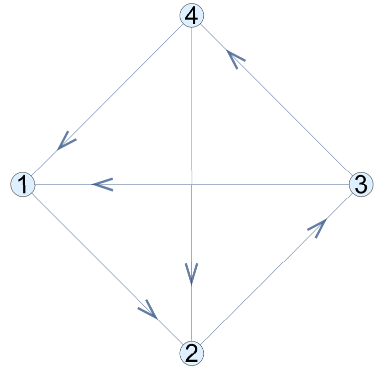
(1) Tetrahedron



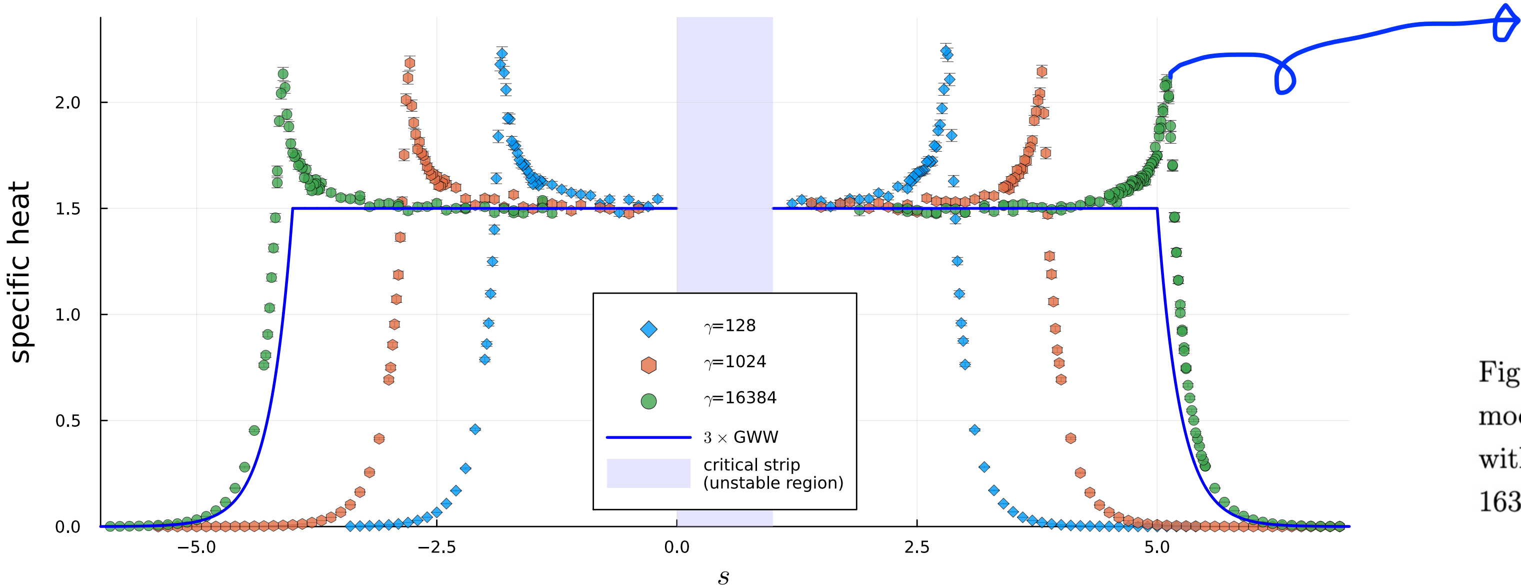
(a) $s = 5.0$ (b) $s = 5.11979$

Figure 12: The eigenvalue distribution of U_1, U_2, U_3 and $U_1U_2U_3$ in the FKM model on K_4 at $s = 5.0$, the phase transition point of the GWW model (a), and at $s = 5.11979$, the value of s closest to the peak of the pointed elbow of the specific heat in Fig. 11 (b).





The shape does not depend on N_c.
3rd-order?



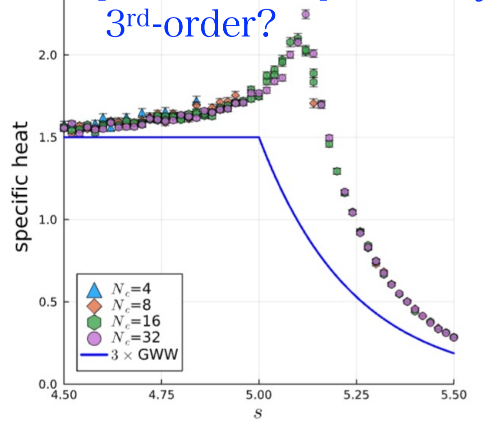
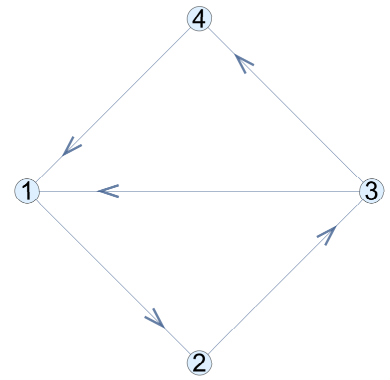
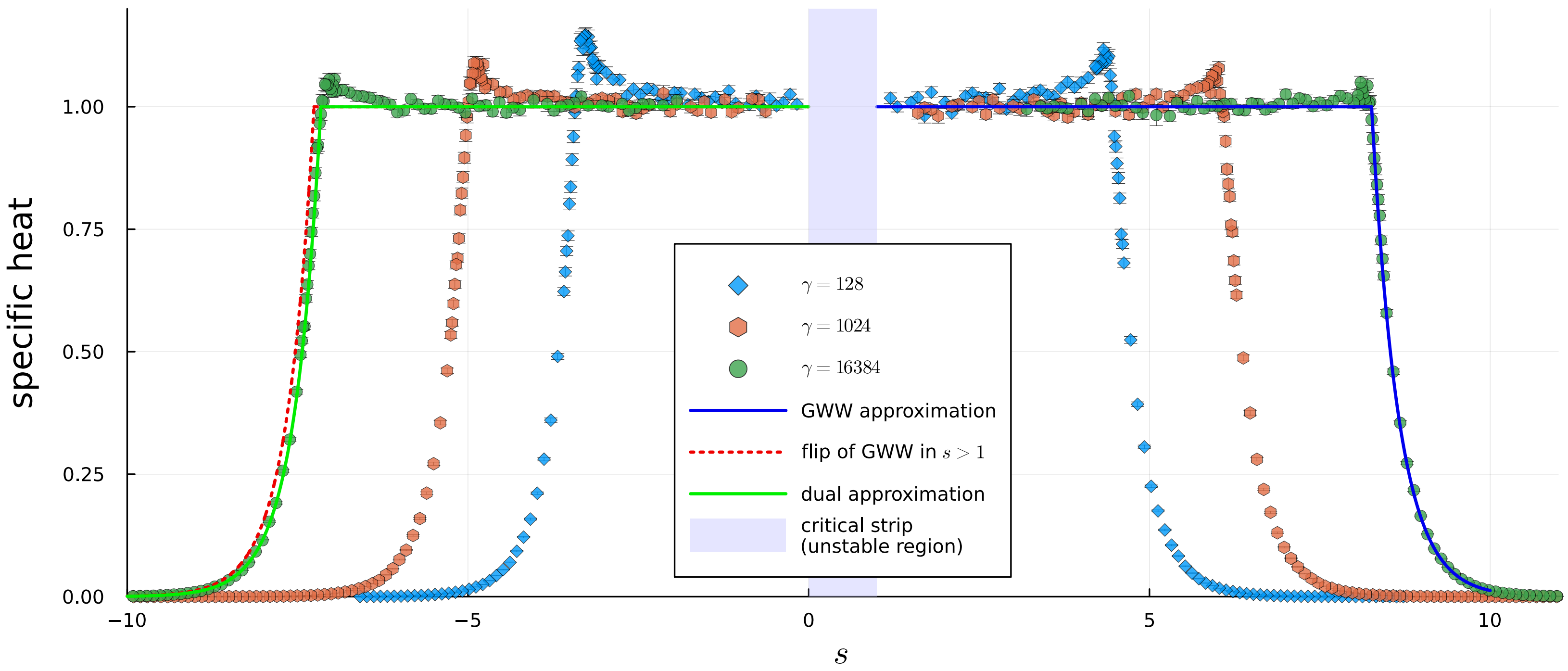


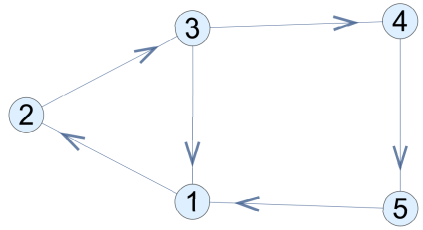
Figure 13: The specific heat of the FKM model on K_4 near the phase transition point with varying $N_c = 4, 8, 16, 32$ and fixing $\gamma = 16384$.

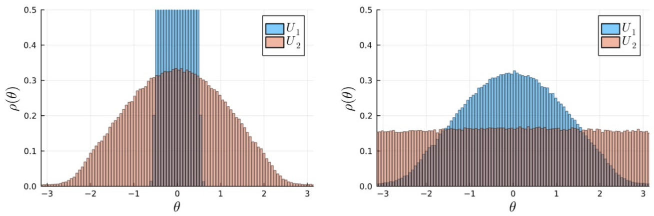
(2) Double Triangle





(3) Triangle-Square

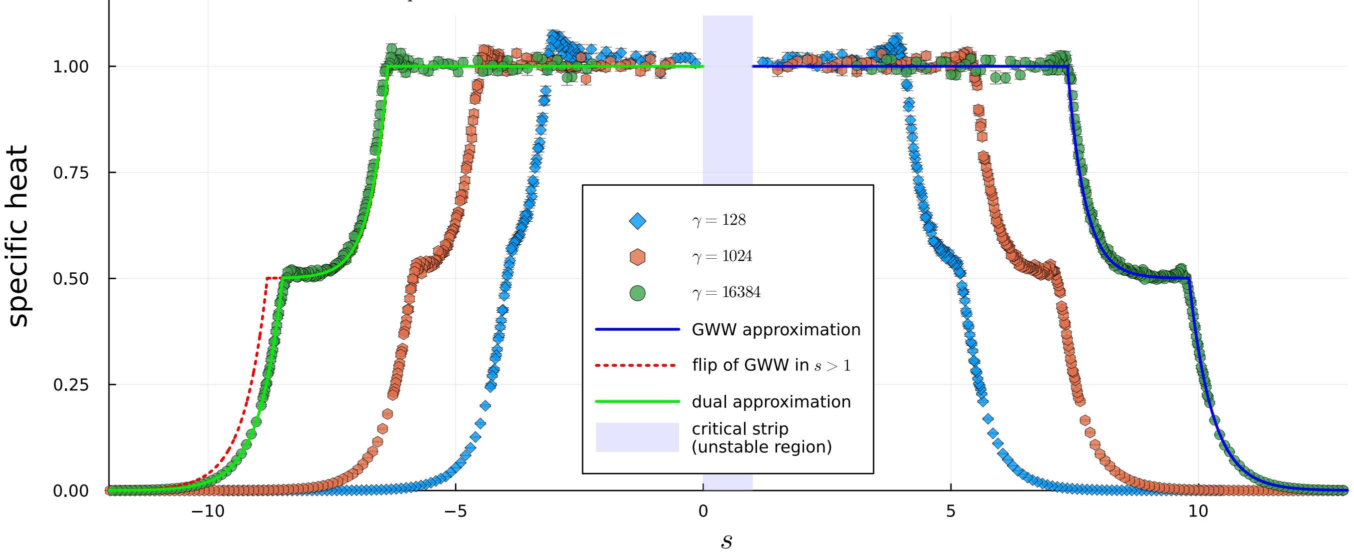




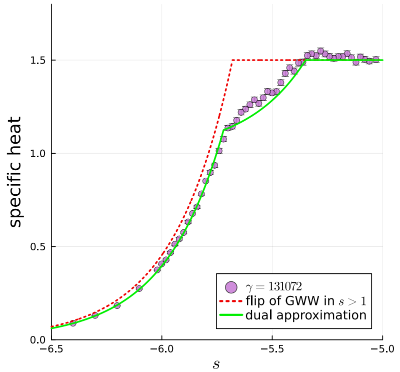
(a) $s = 7.35277$

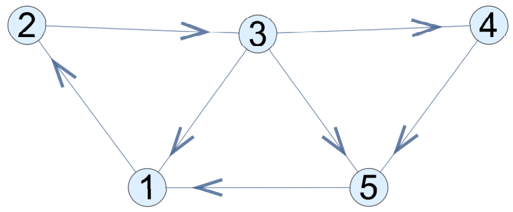
(b) $s = 9.79308$

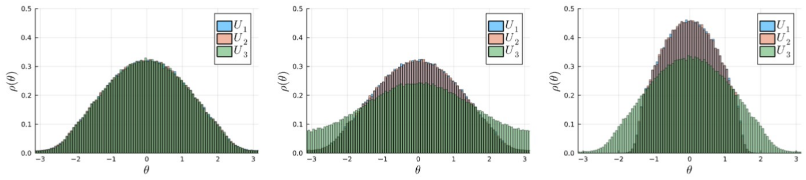
Figure 16: The eigenvalue distributions of U_1 and U_2 at the phase transition points of $N_c = 16$ and $\gamma = 16384$, which are associated with the fundamental cycles of TS with length three and four, respectively. The eigenvalue distribution of U_1 in (a) saturates the upper limit of the figure since it is still too sharp and narrow in the deeper deconfinement phase.



(4) Triple Triangle

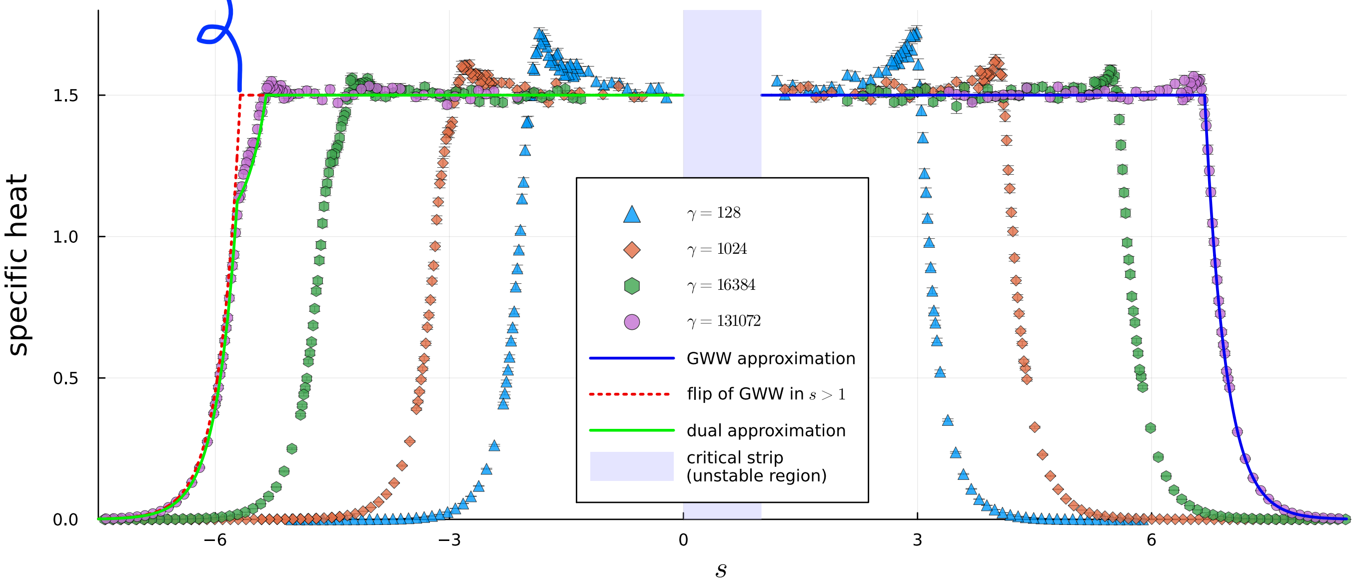






(a) $s = 6.67970$ (b) $s = -5.72085$ (c) $s = -5.34085$

Figure 19: The eigenvalue distributions of the FKM model on TT with $N_c = 16$ and $\gamma = 131072$. (a) The common phase transition point for all unitary matrices in $s > 1$. (b) The phase transition point of U_1 and U_2 in $s < 0$. (c) The phase transition point of U_3 in $s < 0$.



Conclusion

- We constructed the FKM model on the graph and found that the partition function is described by the unitary matrix weighted Ihara zeta function.
- The effective action of the FKM model reduced to the Wilson action in an appropriate parameter limit.
- The FKM model on a regular graph has a strong/weak coupling duality because of the functional equation of the Ihara zeta function.
- The FKM model on an irregular graph has also a dual description.
- The FKM model exhibits the GWW phase transition, and the phase structure depends on the structure of the cycles of the graph.
- We checked the theoretical analysis by numerical simulations.

Future works

- Basic properties of the Bartholdi zeta function are unknown (coming soon)
- 3-matrix model $S = Tr(U_1 + U_2 + U_3 + U_1U_2U_3 + h.c)$ (Does anyone know about this?)
- Continuum limit? or dynamical fermions? (connection to QCD)
- Physical meaning of the Riemann's hypothesis of graph zeta function or Ramanujan graph?
- Can graph zeta function be an observable of SUSY gauge theory on the graph?
- Relation to other zeta functions?

Thank you very much!

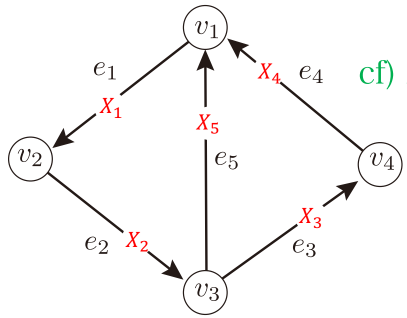
Backups

Matrix weighted Bartholdi zeta function

Ohta-S.M. 2022

cf) Mizuno, Sato 2003,2006

- regular matrix $X_{\boldsymbol{e}}$ (size K) on each edge $\boldsymbol{e}$
- $X_{\boldsymbol{e}^{-1}} = X_{\boldsymbol{e}}^{-1}$
- $X_C \equiv X_{e_{i_1}} \cdots X_{e_{i_n}}$ for $C = e_{i_1} \cdots e_{i_n}$
- matrix weighted adjacency matrices



$$A(X)_{vv'} = \begin{cases} X_e & \langle v, v' \rangle = e \\ 0 & \text{others} \end{cases} \qquad (W_X)_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} X_{\boldsymbol{e}} & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e}') \text{ and } \boldsymbol{e}'^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}, \qquad (J_X)_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} X_{\boldsymbol{e}} & \text{if } \boldsymbol{e}'^{-1} = \boldsymbol{e} \\ 0 & \text{others} \end{cases}.$$

<u>Matrix weighted Bartholdi zeta function</u>

$$\zeta_G(q, u; X) \equiv \prod_{C \in [\mathcal{P}]} \det\left(1_K - q^{|C|} u^{b(C)} X_C\right)^{-1},$$

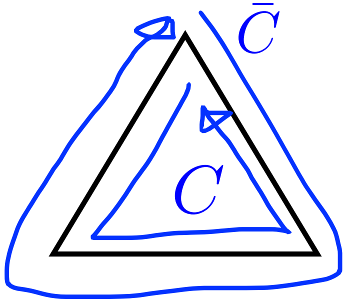
$$= \left(1 - (1-u)^2 q^2\right)^{-K(n_E - n_V)} \det\left(1_{K v_N} - q A_X + (1-u) q^2 (D - (1-u) 1_{K v_N})\right)^{-1}$$

$$= \det\left(1_{2K n_E} - q(W_X + u J_X)\right)^{-1}$$

伊原ゼータ関数の例

primitive reduced cycleはふたつ

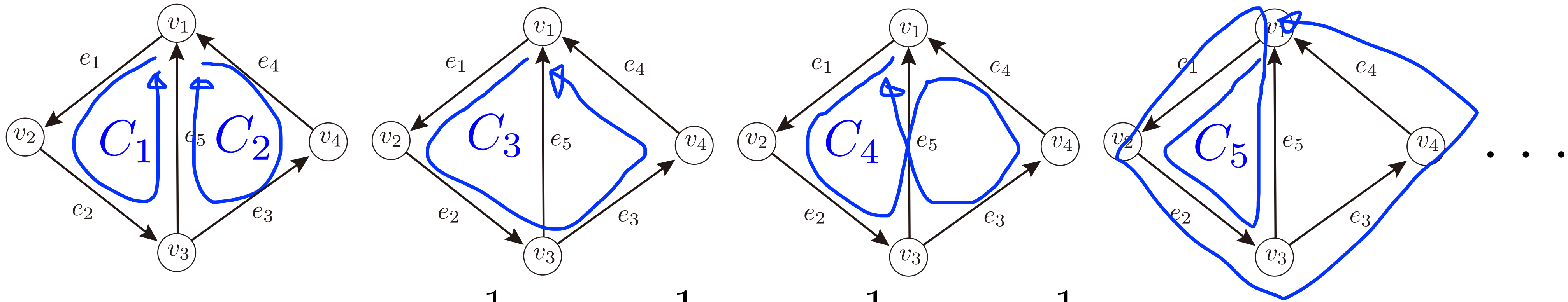
Triangle



$$\zeta_{C_3}(q) = \frac{1}{(1-q^3)^2} = 1 + 2q^3 + 3q^6 + 4q^9 + 5q^{12} + \cdots$$

power of q (length)	3	6	9	12	$\cdots$
coeff	2	3	4	5	$\cdots$
cycles	C, $\bar{C}$	C^2, $C\bar{C}$, $\bar{C}^2$	C^3, $C^2\bar{C}$, $C\bar{C}^2$, $\bar{C}^3$	C^4, $C^3\bar{C}$, $C^2\bar{C}^2$ $C\bar{C}^3$, $\bar{C}^4$	$\cdots$

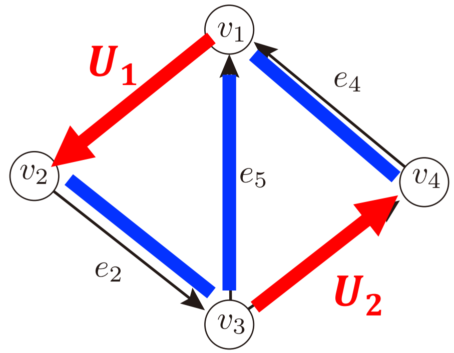
Double Triangle　一般には、primitive reduced cycleは無数にあるため、素朴には閉じた形に書けない



$$\zeta_{\mathrm{DT}}(q) = \frac{1}{(1-q^3)^4}\frac{1}{(1-q^4)^2}\frac{1}{(1-q^6)^2}\frac{1}{(1-q^7)^4}\cdots$$

Gauge fixing and degrees of freedom

$$S_{\text{eff}}(U) = -N_c \log\left(\zeta_G(q;U)\right) = -N_c \log\det\left(1 - qW_U\right)$$



degrees of freedom after gauge fixing

- We can set $U_e = 1$ on a spanning tree
- The number of the remaining edge = **rank r**
- The remaining unitary matrices = independent plaquette variables

$U_1, \cdots, U_r$: fundamental cycles

$$S_{\text{eff}}(U) = -N_C \sum_{[C]} \sum_{n=1}^{\infty} \frac{q^{n|C|}}{n} \left(\operatorname{Tr} U_C^n + \operatorname{Tr} U_C^{-n}\right) \qquad (U_C = U_{a_1} \cdots U_{a_l})$$

Saddle points

$$\delta S_{\mathrm{eff}}(U) = -i \sum_{a=1}^{r} \sum_{C \in R_a} q^{|C|} \operatorname{Tr}\left(\delta A_a \left(U_C - U_C^\dagger\right)\right) = 0$$

reduced cycles including C_a

- $U_C = U_C^\dagger$ for all reduced cycles
- C can be a fundamental cycle: ➡ $U_a = U_a^\dagger$
- For $C = C_a C_b$: $U_a U_b = (U_a U_b)^\dagger = U_b^\dagger U_a^\dagger = U_b U_a$

U_a are diagonalizable simultaneously.

⬇

$$U_a = \mathrm{diag}\,(\pm 1, \cdots, \pm 1)$$

Vacuum and the stability

$$S(U)\Big|_{\mathrm{FP}} = -N_c \sum_{C} \sum_{n=1}^{\infty} \frac{q^{n|C|}}{n} \left(N_C^+ + N_C^- (-1)^n\right)$$

➡ vacuum : $U_a = \mathbf{1}_{N_c}$

<u>The stability of the vacuum</u>

$$S_{\mathrm{eff}}(U) = -N_c^2 \log \zeta_G(q) - \sum_{a,b=1}^{r} \mathrm{Tr}\left(\delta A_a \delta A_b\right) \left(\mathcal{M}_G\right)_{ab} + \mathcal{O}(\delta A^3)$$

<u>Proposal</u>

$\det(\mathcal{M}_G) < 0 \quad 0 < s < 1$

$\det(\mathcal{M}_G) > 0 \quad \text{otherwise}$