Complete implementation of generalized symmetries on the lattice

What is lattice gauge field topology?

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離散的手法による場と時空のダイナミクス2024
@東京工業大学, 3/9/2024

- M. Abe, OM and H. Suzuki, PTEP **2023**, no.2, 023B03 (2023) [2210.12967].
- M. Abe, OM, S. Onoda, H. Suzuki and Y. Tanizaki, JHEP **08**, 118 (2023) [2303.10977].
- OM, Gaugefields.jl, `https://github.com/o-morikawa/Gaugefields.jl`; OM and H. Suzuki, in progress

Symmetry

- Symmetry: fundamental tool in physics
  - Universal applications to high energy physics, condensed matter physics and mathematics
- Noether theorem and conservation law

$$\text{Sym} : \phi \mapsto \phi'; S(\phi) = S(\phi'), \qquad \partial_\mu j^\mu = 0$$

$$\text{Charge } Q \equiv \int j^0 d^{D-1}x$$

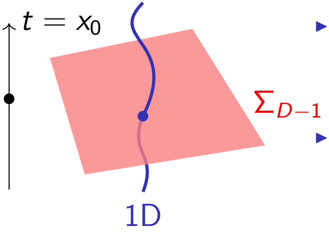
  - Or, a unitary operator commutes with Hamiltonian
- More aspects of Symmetry
  - Landau theory: phase transition (or vacuum structure) from viewpoint of symmetry
  - Symmetry breaking: quantum anomaly, spontaneous breaking

Recent generalization of symmetry

- Generalized the notion of symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]
  - Local description of symmetry is better
- Symmetry described as topological operator
  - Coupled with topological field theory
    - ★ Changing topological structure without changing local dynamics [Kapustin–Seiberg '14]:
    - ★ Nontrivial information from this new kind of symmetry
  - There are some types
    1. Higher-form symmetry (see next slide)
    2. Higher-group symmetry
    3. Non-invertible symmetry
- See reviews, surveys and lectures [Sharpe, Kong–Lan–Wen–Zhang–Zheng, Córdova–Dumitrescu–Intriligator–Shao, McGreevy, Gomes, Schäfer-Nameki, Brennan–Hong, Bhardwaj–Bottini–Fraser-Taliente–Gladden–Gould–Platschorre–Tillim, Luo–Wang–Wang, Shao, Carqueville–Del Zotto–Runkel, Iqbal]

Generalization: higher-form symmetry

- (0-form) Symmetry



  - Charge (codim 1)

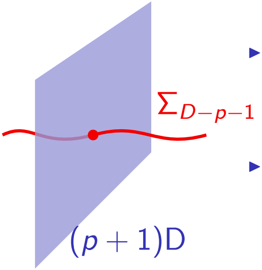
$$Q \equiv \int_{\Sigma_{D-1}} j_0 dx_1 \wedge \cdots \wedge dx_{D-1}$$

  - Symmetry operator

$$U_\alpha(\Sigma_{D-1}) = e^{i\alpha Q}$$

    Topological under deformation of Σ_{D-1}

- Higher-form symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]



  - p-form symmetry $G^{[p]}$ (codim $p+1$)

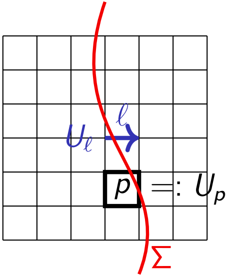
$$Q \equiv \int_{\Sigma_{D-p-1}} \star j^{(p+1)}, \quad U_\alpha(\Sigma) = e^{i\alpha Q}$$

  - Transforming a "loop" operator $W(C)$

$$W(C) \mapsto U(\Sigma)W(C)$$
$$= e^{i\alpha \#(\Sigma, C)} W(C) \quad \text{w/ linking } \#$$

E.g., center symmetry in YM theory

- Lattice $SU(N)$ YM theory
  - link variable $U_\ell \in SU(N)$



- Center symmetry: $\mathbb{Z}_N^{[1]}$

$e^{\frac{2\pi i}{N}k} \in \mathbb{Z}_N \subset SU(N)$; $\quad U_\ell \mapsto e^{\frac{2\pi i}{N}k\#(\Sigma,\ell)} U_\ell$

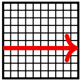
Intersection # of Σ & link ℓ; $U_p \mapsto U_p$

- Gauging the center symmetry

$S \sim \sum \mathrm{Tr}\, e^{-\frac{2\pi i}{N}B_p} U_p \qquad B_p$: 2-form gauge field assoc. $\mathbb{Z}_N^{[1]}$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell,\ B_p \mapsto B_p + (d\lambda)_p$

$\overset{\text{global desc.}}{\Rightarrow}$ Recall 't Hooft twisted b.c. ['79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$



$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N}z_{\mu\nu}} \in \mathbb{Z}_N$

't Hooft flux $z_{\mu\nu} = \sum B_p \bmod N$

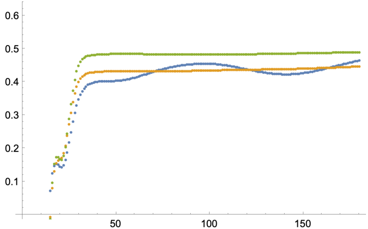
Numerical simulation of higher-form gauge fields

Public repository: `https://github.com/o-morikawa/Gaugefields.jl`

- Implementation of higher-form gauge fields: Gaugefields.jl on GitHub
  - External/dynamical B
  - HMC and gradient flow (MPI)

  based on [LatticeQCD.jl, Gaugefields.jl]
- Under 't Hooft twisted b.c.,
  $Q = \frac{1}{8\pi^2}\int \mathrm{tr}\, F \wedge F \in -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N} + \mathbb{Z}$ [van Baal '82]
- Fractional topological charge for $SU(2)$
  - $Q \sim \frac{1}{8\pi^2 N}\int B \wedge B \in \frac{1}{N}\mathbb{Z}$
  - Compute "Q" with smeared U

  [OM–Suzuki in progress]



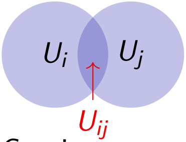
What is topology on the lattice?

- Practically, we observed topology as "Q" $\cong 0.5\ldots$
  - Construction of θ term suffers from failure of odd-ness under time reversal and/or reflection positivity
  - cf. a manifest quantization of winding number [Shiozaki '24]
- Generalized symmetry in continuum
  $\rightarrow$ ('t Hooft) Anomaly between 1-form symmetry & $\theta \sim \theta + 2\pi$
  [Kapustin–Thorngren '13, Gaiotto–Kapustin–Komargodski–Seiberg '17]
- *I believe (all those who join this workshop)*

  You must want to understand it in an ultra-local way,
  that is, within lattice gauge theory.
- Proof:
  - Integer Q
    (4D $SU(N)$) [Lüscher '84], general case [Phillips–Stone '86, '90]
  - Fractional Q for 4D $SU(N)$ coupled with B
    [Abe–OM–Onoda–Suzuki–Tanizaki '23]

Principal fiber bundle

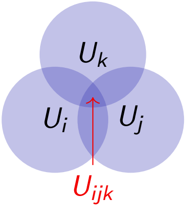
- Recall Dirac's discussion
  - Gauge fields cannot be defined globally in spacetime
  - Defined on each subspace: northern/southern hemisphere
- Manifold (spacetime) X; open covering (patches) $\{U_i\}$
  - Gauge group G, gauge field a_i on U_i
  - Relation between a_i and a_j?



Gauge transformation:

$$a_j = g_{ij}^{-1} a_i g_{ij} + g_{ij}^{-1} d g_{ij} \quad \text{at } U_{ij}$$

  - Consistency condition for transition function g_{ij}:

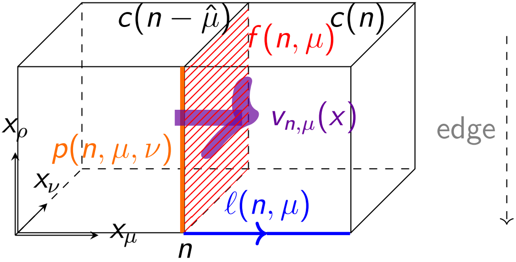


$$g_{ii} = \mathrm{id},\ g_{ji} = g_{ij}^{-1}$$

Cocycle condition: $\quad g_{ij} g_{jk} g_{ki} = 1 \quad$ at U_{ijk}

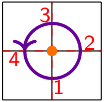
Bundle structure on lattice?

- No continuity for lattice fields?
  - Any configuration can be deformed continuously to others
  - We can observe topological structure even on lattice by Lüscher
- Setup/strategy: Lattice Λ divides X into 4D hypercubes



4D cell $c(n)$
3D face $f(n,\mu)$
2D plaquette $p(n,\mu,\nu)$
1D link $\ell(n,\mu)$

1. Regard $\{c(n)\}$ as patches
2. Define transition function $v_{n,\mu}(x)$ at $f(n,\mu)$ from data as U_ℓ
   - Difficult to define it at $x \neq n$ s.t. cocycle condition is kept intact



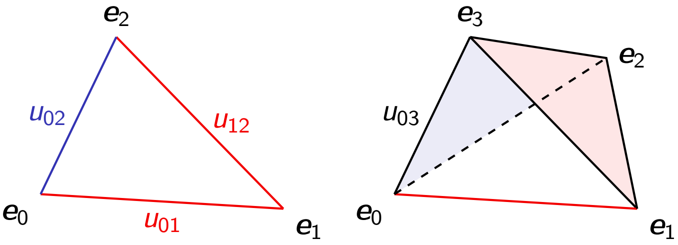
$v_1 v_2 v_3^{-1} v_4^{-1} = 1$ at $x \in p(n)$

Definition of parallel transport function

- At first, let's illustrate an iterative method on nD $GL(p, \mathbb{C})$ simplicial lattice (triangle) [Phillips–Stone '86, '90]
- Let Δ^r be a r-simplex in X; parallel transport function V as

$$V_{\langle 0\dots r\rangle}(s_1, \dots, s_p = 1, \dots, s_{r-1}) = V_{\langle 0\dots p\rangle}(s_1, \dots, s_{p-1}) \cdot V_{\langle p\dots r\rangle}(s_{p+1}, \dots, s_{r-1}),$$

$$V_{\langle 0\dots r\rangle}(s_1, \dots, s_p = 0, \dots, s_{r-1}) = V_{\langle 0\dots \hat{p}\dots r\rangle}(s_1, \dots, \hat{s}_p, \dots, s_{r-1})$$

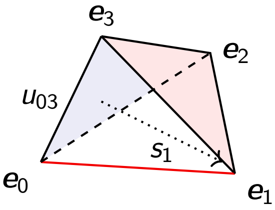


$V(0) = u_{02}$
$V(1) = u_{01}u_{12}$

$V(0) = u_{02}u_{23}$
$V(1) = u_{01}u_{12}u_{23}$

...

Reconstruction of principal G-bundle



- Successive linear interpolation

$$V_{\langle 0\ldots r\rangle}(\ldots, s_{r-1}) = (1 - s_{r-1})\, V_{\langle 0\ldots r-2,r\rangle} + s_{r-1}\, V_{\langle 0\ldots,r-1\rangle}\, u_{r-1,r}$$

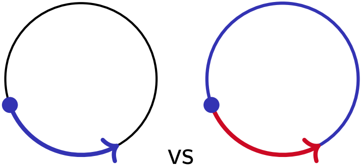
  - This is for $GL(p, \mathbb{C})$
  - For other G, we need a projection or geodesic approach
    - ★ Recall $SU(3)$ projection in conf generation in lattice simulation
- (Math) Theorem
  - Projection $\pi : E \to M$, section $H : M \to E$ s.t. $\pi \circ H = \mathrm{id}_M$
  - Definition of H

1. if $\sigma = \langle i \rangle$, $H_\sigma : \mathbf{0} \mapsto \boldsymbol{e}_i$
2. $H_{\langle 0\ldots r\rangle}(\ldots, s_r) = (1 - s_r) H_{\langle 0\ldots r-1\rangle} + s_r V_{\langle 0\ldots r\rangle} \boldsymbol{e}_r$

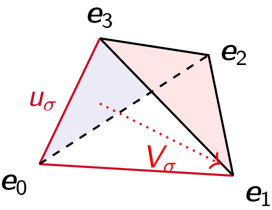
  - Define $\Sigma_q = \{A | \mathrm{rank}(A^1, \ldots, A^{p-q+1}) \leq p - q\}$
  - qth Chern class is represented by "intersection" of H_σ and Σ_q

Is $V_\sigma(\forall s)$ compatible with u_σ?

- How different are $V_\sigma(\forall s)$ and u_σ?
  - $u_\sigma = u_{ij}$ where $\sigma = \langle i \dots j \rangle$



- For simplicity, e.g., in the $U(1)$ case,
- $\exists V(s)$ s.t. $||V_\sigma - u_\sigma|| < \delta\varphi(K,n)$ for $\forall s$
  - δ: a parameter
  - K is maximum of operator norm
  - n dimensions
  - $\varphi(K,n)$ is determined by $\sim K^r$, $r \le n$



- Then, $\exists\epsilon$ s.t. if $||u - u_0|| < \epsilon$
  u determines the same bundle as u_0
- [NOTE] *GL* group makes the above statement simpler.
  - These expressions were given by Phillips–Stone
  - In general, *explicitly* gauge invariant forms are suitable

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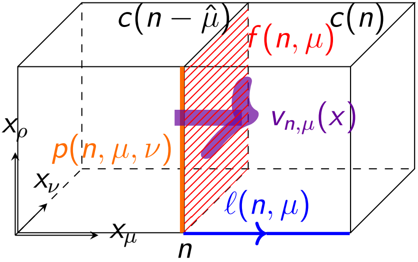

edge

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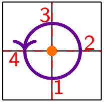


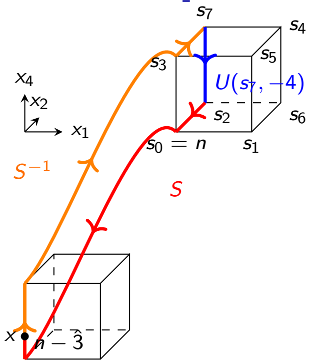
$v_1 v_2 v_3^{-1} v_4^{-1} = 1$ at $x \in p(n)$

Construction of topo. sectors on lattice [Lüscher]

- *Parallel transporter* (interpolation):
  $U_\ell \to v_f(n) \to v_f(x) \Leftarrow$ Cocycle

  $v_{n,\mu}(n) = U(n-\hat{\mu},\mu)$

  $v_{n,\mu}(x) \equiv S^{n-\hat{\mu}}_{n,\mu}(x)^{-1} v_{n,\mu}(n) S^{n}_{n,\mu}(x)$



- Topo. sectors on lattice so that $[Q$ in terms of $v_f(x)] \in \mathbb{Z}$

$$Q = \sum_{n\in\Lambda} q(n), \quad q(n) = -\frac{1}{24\pi^2}\sum_{\mu,\nu,\rho,\sigma}\epsilon_{\mu\nu\rho\sigma}\left\{3\int_{p_{n,\mu\nu}} d^2x \ \mathrm{Tr}\left[\left(v_{n,\mu}\partial_\rho v^{-1}_{n,\mu}\right)\left(v^{-1}_{n-\hat{\mu},\nu}\partial_\sigma v_{n-\hat{\mu},\nu}\right)\right]\right.$$

$$\left. + \int_{f_{n,\mu}} d^3x \ \mathrm{Tr}\left[\left(v^{-1}_{n,\mu}\partial_\nu v_{n,\mu}\right)\left(v^{-1}_{n,\mu}\partial_\rho v_{n,\mu}\right)\left(v^{-1}_{n,\mu}\partial_\sigma v_{n,\mu}\right)\right]\right\}$$

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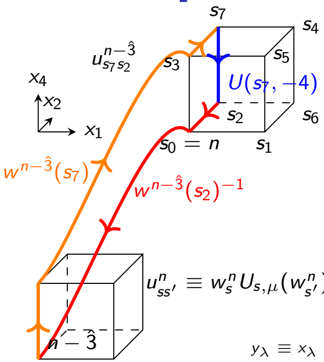


$$u^{n}_{ss'} \equiv w^{n}_{s} U_{s,\mu} (w^{n}_{s'})^{-1}$$

$$y_\lambda \equiv x_\lambda - n_\lambda$$

$$w^n(x) = U^{y_4}_{n,4} U^{y_3}_{n+y_4\hat{4},3} U^{y_2}_{n+y_4\hat{4}+y_3\hat{3},2} U^{y_1}_{n+y_4\hat{4}+y_3\hat{3}+y_2\hat{2},1}$$

$$f^m_{n,\mu}(x_\gamma) = (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_7} u^m_{s_7 s_2} u^m_{s_2 s_0})^{y_\gamma} u^m_{s_0 s_2} (u^m_{s_2 s_7})^{y_\gamma},$$

$$g^m_{n,\mu}(x_\gamma) = (u^m_{s_5 s_1})^{y_\gamma} (u^m_{s_1 s_5} u^m_{s_5 s_4} u^m_{s_4 s_6} u^m_{s_6 s_1})^{y_\gamma} u^m_{s_1 s_6} (u^m_{s_6 s_4})^{y_\gamma},$$

$$h^m_{n,\mu}(x_\gamma) = (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_5} u^m_{s_5 s_1} u^m_{s_1 s_0})^{y_\gamma} u^m_{s_0 s_1} (u^m_{s_1 s_5})^{y_\gamma},$$

$$k^m_{n,\mu}(x_\gamma) = (u^m_{s_7 s_2})^{y_\gamma} (u^m_{s_2 s_7} u^m_{s_7 s_4} u^m_{s_4 s_6} u^m_{s_6 s_2})^{y_\gamma} u^m_{s_2 s_6} (u^m_{s_6 s_4})^{y_\gamma},$$

$$l^m_{n,\mu}(x_\beta, x_\gamma) = [f^m_{n,\mu}(x_\gamma)^{-1}]^{y_\beta} [f^m_{n,\mu}(x_\gamma) k^m_{n,\mu}(x_\gamma) g^m_{n,\mu}(x_\gamma)^{-1} h^m_{n,\mu}(x_\gamma)^{-1}]^{y_\beta} h^m_{n,\mu}(x_\gamma) [g^m_{n,\mu}(x_\gamma)]^{y_\beta},$$

$$S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) = (u^m_{s_0 s_3})^{y_\gamma} [f^m_{n,\mu}(x_\gamma)]^{y_\beta} [l^m_{n,\mu}(x_\beta, x_\gamma)]^{y_\alpha}.$$

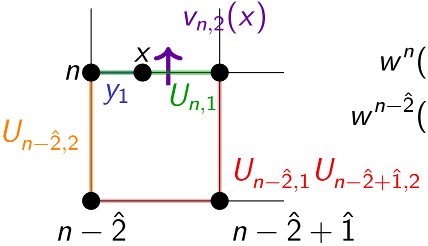
- Topo. sectors on lattice so that [Q in terms of $v_f(x)$] $\in \mathbb{Z}$

$$Q = \sum_{n\in\Lambda} q(n), \quad q(n) = -\frac{1}{24\pi^2} \sum_{\mu,\nu,\rho,\sigma} \epsilon_{\mu\nu\rho\sigma} \Big\{ 3 \int_{p_{n,\mu\nu}} d^2x \, \mathrm{Tr}\Big[\left(v_{n,\mu}\partial_\rho v^{-1}_{n,\mu}\right) \left(v^{-1}_{n-\hat{\mu},\nu}\partial_\sigma v_{n-\hat{\mu},\nu}\right) \Big]$$

$$+ \int_{f_{n,\mu}} d^3x \, \mathrm{Tr}\Big[\left(v^{-1}_{n,\mu}\partial_\nu v_{n,\mu}\right) \left(v^{-1}_{n,\mu}\partial_\rho v_{n,\mu}\right) \left(v^{-1}_{n,\mu}\partial_\sigma v_{n,\mu}\right) \Big] \Big\}$$

Exercise: Lattice 2D $U(1)$ gauge theory

- $v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^n(x)^{-1}$ [Lüscher '98, Fujiwara et al. '00]



$$w^n(x) = U_{n,1}{}^{y_1}$$
$$w^{n-\hat{2}}(x) = [U_{n-\hat{2},1} U_{n-\hat{2}+\hat{1},2}]^{y_1} U_{n-\hat{2},2}{}^{1-y_1}$$

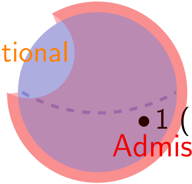
- Explicit expression of v:

$$v_{n,1}(x) = U_{n-\hat{1},1} \quad v_{n,2}(x) = U_{n-\hat{2},2} [U_{n-\hat{2},1} U_{n-\hat{2}+\hat{1},2} U_{n,1}^{-1} U_{n-\hat{2},2}^{-1}]^{y_1}$$
$$= U_{n-\hat{2},2} \exp[i y_1 F_{12}(n-\hat{2})]$$

  - Field strength: $F_p \equiv \frac{1}{i} \ln U_p$ for $-\pi < F_p \leq \pi$
  - (nD) To ensure Bianchi identity $dF_p = 0$, we should impose $\sup_p |F_p| < \epsilon$, $0 < \epsilon < \frac{\pi}{3}$ → Admissibility condition

Admissibility condition

- In general, admissibility = well-defined-ness of u^y $(0 \le y \le 1)$
  - ★ $U(1)$: $F_p = \frac{1}{i} \ln U_p$ for plaquette U_p
  - ★ $S^m_{n,\mu}(x)$ is written in terms of $(u^n_{ss'})^y$ where u is a loop $n \to s \to s' \to n$
  - ▸ E.g., u^y is ill-defined at $u = -1$; ill-def regions separate sectors
- Admissibility condition $\mathrm{tr}(1 - U_p) < \epsilon$ [Lüscher '84]



- ▸ Admissible lattice gauge fields: well-defined conf space ∼ disk
- ▸ Exceptional region
  - ★ Topological freezing
  - ★ Monopole as lattice artifact

- Under the admissibility condition, we can prove that $Q \in \mathbb{Z}$; we observe topo. sectors even on lattice!
- How about index theorem for finite a?

$$\mathrm{Index}(D) = \underbrace{-\frac{a}{2}\mathrm{Tr}\,\gamma_5 D_{\mathrm{ov}}}_{\text{Admissibility } \epsilon_{\mathrm{ov}}} = \underbrace{n_+ - n_-}_{\in\mathbb{Z}} \overset{?}{=} \frac{1}{32\pi^2}\int_x \varepsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}[F_{\mu\nu}F_{\rho\sigma}]$$

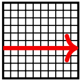
Cocycle condition relaxed by higher-form sym

- Lüscher's construction *Coupled* with higher-form gauge fields
  - $SU(N)$ YM theory coupled with $\mathbb{Z}_N$ 2-form gauge field

$$S \sim \sum \mathrm{Tr}\, e^{-\frac{2\pi i}{N} B_p} U_p \qquad B_p\text{: 2-form gauge field assoc. } \mathbb{Z}_N^{[1]}$$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell,\ B_p \mapsto B_p + (d\lambda)_p$

  - 't Hooft twisted b.c. ['79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$

0  gauge transf L

$$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$$

't Hooft flux $z_{\mu\nu} = \sum B_p \bmod N$

- Cocycle condition can take a $\mathbb{Z}_N$ value:

$$\tilde{v}_{n-\hat{\nu},\mu}(x)\tilde{v}_{n,\nu}(x)\tilde{v}_{n,\mu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = e^{\frac{2\pi i}{N} B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}$$

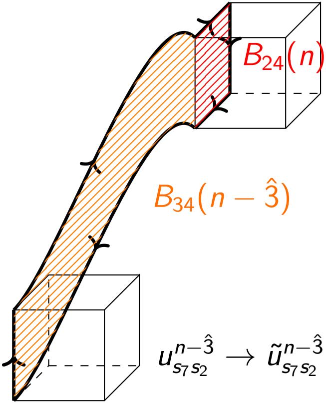
  - $\mathbb{Z}_N$ blind matters: adjoint repr.
  - $\mathbb{Z}_N^{[1]}$ gauge inv. if $\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N}\lambda_{n-\hat{\mu},\mu}} \tilde{v}_{n,\mu}(x)$
- What is definition of $\tilde{v}_{n,\mu}(x)$?

$\mathbb{Z}_N^{[1]}$ gauge invariant construction

- Gauge invariant plaquette
  $\tilde{U}_p \equiv e^{-\frac{2\pi i}{N}B_p}U_p$
  - Recall $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell}U_\ell$
  - Admissibility $\mathrm{tr}(1-\tilde{U}_p) < \epsilon$
- u: product of plaquettes $\to \tilde{u}$

$$\tilde{u}^{n-\hat{3}}_{s_7 s_2} = e^{\frac{2\pi i}{N}B_{34}(n-\hat{3})}e^{\frac{2\pi i}{N}B_{24}(n)}u^{n-\hat{3}}_{s_7 s_2}$$

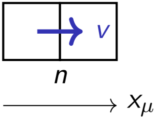
- Similarly, $\tilde{v}$ is defined in terms of $\tilde{u}$
  - Gauge covariance
    $\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N}\lambda_\mu(n-\hat{\mu})}\tilde{v}_{n,\mu}(x)$



- Fractional topological charge $Q = \sum_n q(n) \in -\frac{\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}}{8N} + \mathbb{Z}$
- Implementation of B_p is clear now!
  - `https://github.com/o-morikawa/Gaugefields.jl`

Summary

- Generalized symmetries have been developed in this decade
  - Higher-form sym, higher-group sym, noninvertible sym, subsystem sym, ...
  - Through the use of 't Hooft anomaly matching, new insights about *nontrivial dynamics & classification of phases*
- Standing on a fully regularized framework: lattice gauge theory
  - Generalization of Lüscher's construction of topology on lattice
  - Maintaining locality, $SU(N)$ gauge inv & higher-form gauge inv
  - There exists interpolation to smooth enough bundle structure
    - ★ Transition function $v_f(n) \to v_f(x)$
    - ★ Q is written in terms of $v_f(x)^{-1}\partial_\nu v_f(x)$



$$Q \in \mathbb{Z} \xrightarrow{\text{Gauging } \mathbb{Z}_N^{[1]}} \frac{1}{N}\mathbb{Z}$$

  - Mixed 't Hooft anomaly between $\mathbb{Z}_N^{[1]}$&θ periodicity

7 't Hooft anomaly

Backup: 't Hooft anomaly matching

- Problem: Nontrivial dynamics of strongly coupled theories
- Global symmetry may tell us about something ['t Hooft '79]
  - Assume global symmetry G in system
  - Introduce background gauge field A assoc. G (by gauging G)

$$\mathcal{Z}[A] = \int \mathcal{D}\{\phi\}\, e^{-S(\{\phi\},A)}$$

$$\stackrel{?}{=} \mathcal{Z}[A^g] = \cdots = e^{\mathcal{A}[A,g]}\mathcal{Z}[A]$$

$e^{\mathcal{A}} \neq 1 \xrightarrow{\text{Anomalous}}$ This is called 't Hooft anomaly

- 't Hooft anomaly is invariant at any energy scale (renormalization group inv.)
- Restriction on low-energy dynamics: SSB, phase structure, SPT

Backup: 't Hooft anomaly (1-form sym & θ)

- Topological objects from lattice viewpoint as center sym
- Formal discussion in continuum theory
  - $\mathbb{Z}_N^{[q]}$ gauge field: $U(1)$ fields $B^{(q)}$, $B^{(q-1)}$
  - Constraint: $NB^{(q)} = dB^{(q-1)}$
  - This implies charge-N Higgs; breaking as $U(1) \to \mathbb{Z}_N$
- $Q \sim \frac{1}{N} \int B \wedge B? \xrightarrow[\text{cohomology op}]{\text{Swapping} \wedge \text{ w/}} \frac{1}{N} \int P_2(B) \sim -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N}$
  - Global nature described by Čech cohomology (discrete group!)
- Indicating mixed 't Hooft anomaly with chiral sym/θ-periodicity

$$\mathcal{Z}_{\theta+2\pi}[B_p] = e^{-2\pi i Q} \mathcal{Z}_\theta[B_p]$$
$$\neq \mathcal{Z}_\theta[B_p] \qquad \text{not } 2\pi \text{ periodic}$$