ローレンツ対称性を保つタイプIIB行列模型の新しい定義とその必要性

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Ref.) Asano, JN, Piensuk, Yamamori, arXiv:2404. 14045
Asano, JN, Piensuk, Yamamori, in preparation
Chou, JN, Tripathi, in preparation

0. Introduction

type IIB (or IKKT) matrix model

Ishibashi-Kawai-Kitazawa-Tsuchiya,
Nucl.Phys.B 498 (1997) 467, hep-th/9612115 [hep-th]

- a nonperturbative formulation of superstring theory
  ``lattice gauge theory'' of everything (matter, force and space-time)

$$
\begin{aligned}
S_{\mathrm{b}} &= -\frac{1}{4} N \,\mathrm{tr}([A_\mu, A_\nu][A^\mu, A^\nu]) \\
S_{\mathrm{f}} &= -\frac{1}{2} N \,\mathrm{tr}(\Psi_\alpha (\mathcal{C}\,\Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta])
\end{aligned}
$$

(0-dimensional reduction of 4D $\mathcal{N}=4$ SYM)

$N \times N$ Hermitian matrices

SO(9,1) Lorentz symmetry

A_μ $(\mu = 0, \cdots, 9)$ Lorentz vector

Ψ_α $(\alpha = 1, \cdots, 16)$ Majorana-Weyl spinor

Lorentzian metric $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$ is used to raise and lower indices.

- Unlike AdS/CFT, not only space but also time emerge as the eigenvalue distribution of the 10 bosonic matrices.

maximal SUSY (incl. translation : $A_\mu \mapsto A_\mu + \alpha_\mu \mathbf{1}$)

classical solutions

- Eq. of motion : $[A^\nu, [A_\nu, A_\mu]] = 0$

Classical solutions are exhausted by the diagonal ones ($[A_\mu, A_\nu] = 0$)

See Appendix A of
H. C. Steinacker, JHEP 02, 033,
arXiv:1709.10480 [hep-th].

Add a Lorentz invariant “mass” term to the IKKT action.

$$S_\mathrm{m} \;=\; -\frac{1}{2} N\gamma \,\mathrm{tr}(A_\mu A^\mu) = \frac{1}{2} N\gamma \left\{\mathrm{tr}(A_0)^2 - \mathrm{tr}(A_i)^2\right\}$$

$$S_\mathrm{b} \;=\; \frac{1}{4} N \,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4} N \left\{-2\,\mathrm{tr}(F_{01})^2 + \mathrm{tr}(F_{ij})^2\right\}$$

$F_{\mu\nu} = i\,[A_\mu, A_\nu]$
(Hermitian)

Eq. of motion : $[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0$

Many classical solutions representing expanding space-time appear.

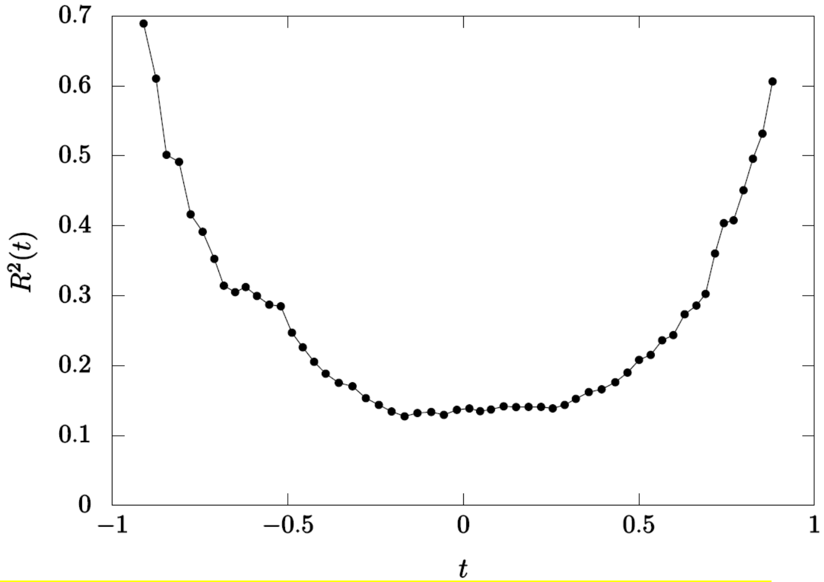
Kim-J.N.-Tsuchiya, 1208.0711
Sperling-Steinacker 1901.03522

typical classical solutions

Eq. of motion : $[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0$

Hatakeyama-Matsumoto-J.N.-
Tsuchiya-Yosprakob,
PTEP 2020 (2020) 4, 043B10

- $A_\mu = 0$ is always a solution.
  (trivial solution)
- Typical Hermitian A_μ solutions
  show expanding behavior for $\gamma > 0$
  but **not** for $\gamma < 0$!
- However, space-time dimensionality
  is not determined at the classical level.



We have to investigate the partition function including the effects of fermions in the 1) $N \to \infty$, 2) $\gamma \to +0$ lim. to see if (3+1)D expanding space-time appears.

- $\gamma = 0$ is a "strong coupling" limit.

$$Z = \int dA\, e^{i(A^4 + \gamma A^2)} \qquad A_\mu = \sqrt{|\gamma|}\tilde{A}_\mu$$

$$= \int dA\, e^{i\gamma^2(\tilde{A}^4 + \tilde{A}^2)} \qquad \gamma^2 \Leftrightarrow \frac{1}{\hbar}$$

Quantum effects become important.
The role of SUSY.

partition function of the type IIB matrix model

$$
\begin{aligned}
Z &= \int dA\, d\Psi\, e^{i(S_\mathrm{b}+S_\mathrm{m}+S_\mathrm{f})} \\
&= \int dA\, e^{i(S_\mathrm{b}+S_\mathrm{m})}\, \underbrace{\mathrm{Pf}\mathcal{M}(A)}_{\text{polynomial in } A}
\end{aligned}
$$

pure phase factor

<u>The partition function is NOT absolutely convergent.</u>

- As a regularization, it was proposed to add convergence factors.

$$Z = \int dA\, e^{i(S_\mathrm{b}+S_\mathrm{m})}\, \mathrm{Pf}\mathcal{M}(A)$$

Anagnostopoulos-Azuma-Hatakeyama-Hirasawa-J.N.-Papadoudis-Tsuchiya, in preparation

$$S_\mathrm{m}^{(\varepsilon)} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon}\mathrm{tr}(A_0)^2 - e^{-i\varepsilon}\mathrm{tr}(A_i)^2 \right\}$$

This breaks Lorentz symmetry!

In fact, the partition function diverges in the $\varepsilon \to 0$ limit due to noncompact flat directions, and the cutoff artifact remains.

Asano, JN, Piensuk, Yamamori, in preparation

What we do in this talk

- We study N=2 bosonic model with the cutoff nonperturbatively by 1/D expansion and the generalized thimble method (GTM).

- In particular, “classicalization” occurs in the $\epsilon \to 0$ limit due to an artifact of the Lorentz symmetry breaking cutoff.

⇓

- We propose a new definition of the type IIB matrix model that respects Lorentz symmetry using Faddeev-Popov gauge fixing.

- Our results for the gauge-fixed model (obtained by GTM) show very different behaviors.

Once we understand the N=2 bosonic model completely,
we just have to do the same things for larger N with SUSY.

Plan of the talk

0. Introduction
1. The need for “gauge-fixing” in the toy model
2. Classical solutions in N=2 bosonic model
3. 1/D expansion and MC studies of the cutoff model
4. MC studies of the “gauge-fixed” model
5. Summary and discussions

1. The need for gauge fixing in the toy model

Asano, JN, Piensuk, Yamamori, arXiv:2404. 14045

a toy model with Lorentz symmetry

type IIB matrix model

$$A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a \enclose{circle}{t^a}$$

basis of traceless Hermitian matrices

a model of (N^2-1) Lorentz vectors

a toy model with a single Lorentz vector x_μ $(\mu = 0, 1, \cdots, d)$

$$Z = \int dx\, e^{-\frac{1}{2}\gamma(x_\mu x^\mu + 1)^2} \qquad \gamma > 0$$

Saddle points : (i) $(x_0)^2 - (x_i)^2 = 1$ ➡ Saddle points of this type are related with each other by Lorentz transformation.

(ii) $x_\mu = 0$

Z diverges due to flat directions

- cutoff model

$$Z_\varepsilon = \int dx\, e^{-\frac{1}{2}\gamma(x_\mu x^\mu + 1)^2 - \varepsilon(x_0)^2 - \varepsilon(x_i)^2} \qquad \lim_{\varepsilon \to 0} Z_\varepsilon = \infty$$

classicalization in the cutoff model

- cutoff model

$$Z_\varepsilon = \int dx\, e^{-\frac{1}{2}\gamma(x_\mu x^\mu+1)^2-\varepsilon(x_0)^2-\varepsilon(x_i)^2}$$

- One can solve this model by introducing an auxiliary variable k

$$Z_\epsilon = \frac{1}{\sqrt{2\pi\gamma}}\int dk\, dx\, e^{-\frac{1}{2\gamma}k^2+ik(x_\mu x^\mu+1)-\epsilon(x_0)^2-\epsilon(x_i)^2}$$

$$Z_\epsilon = \frac{1}{\sqrt{2\pi\gamma}}\int dk\, e^{-\frac{1}{2\gamma}k^2+ik}\sqrt{\frac{\pi}{ik+\epsilon}}\left(\sqrt{\frac{\pi}{-ik+\epsilon}}\right)^d$$

$$= \mathcal{N}\int dk\, e^{-S_{\text{eff}}(k)}$$

$$S_{\text{eff}}(k) = \frac{1}{2\gamma}k^2 - ik + \frac{1}{2}\log(ik+\epsilon) + \frac{d}{2}\log(-ik+\epsilon)$$

$$0 = \frac{dS_{\text{eff}}(k)}{dk} = \frac{1}{\gamma}k - i + \frac{i}{2}\frac{1}{ik+\epsilon} - \frac{id}{2}\frac{1}{-ik+\epsilon}$$

dominant saddle point

$$k^{(0)} \simeq i\frac{d-1}{d+1}\epsilon + i\frac{8d}{(d+1)^3}\epsilon^2 + \mathrm{O}(\epsilon^3)$$

$$Z_\epsilon \sim \epsilon^{-\frac{d+1}{2}}(\times\epsilon) \sim \epsilon^{-\frac{d-1}{2}}$$

Partition function diverges for $\epsilon \to 0$

$$\boxed{\langle k\rangle_\epsilon} = i\gamma\langle(x_\mu x^\nu + 1)\rangle_\epsilon$$

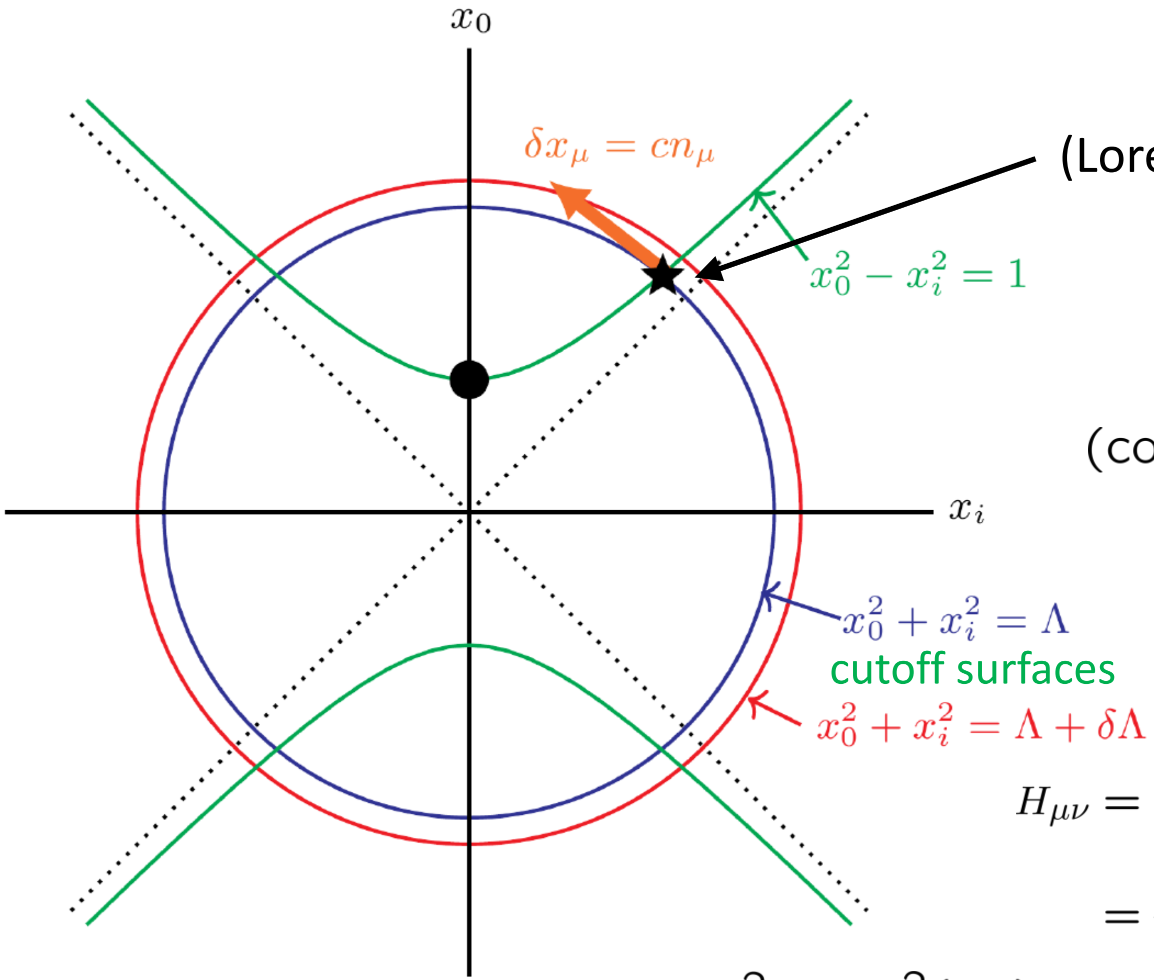
$$\Big\downarrow\ \epsilon \to 0$$

$$0$$

$$-\lim_{\varepsilon\to 0}\langle x_\mu x^\mu\rangle_\varepsilon = 1$$

classicalization

classicalization : an artifact of the cutoff



(Lorentz boosted) saddle point

$$(x_0, x_1) = (\cosh\sigma, \sinh\sigma)$$

$$(\cosh\sigma)\delta x_0 + (\sinh\sigma)\delta x_1 = 0$$

$$\begin{pmatrix} \delta x_0 \\ \delta x_1 \end{pmatrix} = \frac{\textcircled{c}}{\sqrt{\cosh 2\sigma}} \begin{pmatrix} \sinh\sigma \\ -\cosh\sigma \end{pmatrix}$$

magnitude of the fluctuations

$$H_{\mu\nu} = \frac{\partial^2 S(x)}{\partial x_\mu \partial x_\nu} = \gamma\, x^\mu x^\nu$$

$$= \gamma \begin{pmatrix} \cosh^2\sigma & -\cosh\sigma\sinh\sigma \\ -\cosh\sigma\sinh\sigma & \sinh^2\sigma \end{pmatrix}$$

$$\delta S = \delta x_\mu H_{\mu\nu}(x) \delta x_\nu = \frac{c^2\gamma\sinh^2(2\sigma)}{\cosh(2\sigma)}$$

➡ $|c| \lesssim e^{-\sigma}\sqrt{\frac{2}{\gamma}}$ at large σ

Classicalization occurs
due to the cutoff artifact!

“gauge-fixing” the Lorenz symmetry

- “Gauge fixing” condition : minimize: $(x_0)^2$ w.r.t. Lorentz tr.

$$\begin{pmatrix} x'_0 \\ x'_j \end{pmatrix} = \begin{pmatrix} \cosh\sigma & \sinh\sigma \\ \sinh\sigma & \cosh\sigma \end{pmatrix} \begin{pmatrix} x_0 \\ x_j \end{pmatrix}$$

$x_0 x_j = 0$ for all j

$$Z = \int dx \, e^{-\frac{1}{2}\gamma(x_\mu x^\mu + 1)^2} \Delta_{\mathrm{FP}}[x] \prod_{j=1}^{d} \delta(x_0 x_j)$$

$$\Delta_{\mathrm{FP}}[x] = \det\Omega \ , \quad \Omega_{ij} = (x_0)^2 \delta_{ij} + x_i x_j$$

- In fact, the time-like region dominates over the space-like region.

$(x_j = 0)$ $(x_0 = 0)$

$$Z_{\mathrm{g.f.}} = \int dx_0 \, |x_0|^d e^{-\frac{1}{2}\gamma\{-(x_0)^2+1\}^2}$$

$$-\langle x_\mu x^\mu \rangle = 1 + \frac{d-1}{2\gamma} + \cdots$$

Fluctuations exist for finite γ

The classicalization in the cutoff model is an artifact
of Lorentz symmetry breaking that remains in the $\varepsilon \to 0$ limit.

a new definition of type IIB matrix model

- "Gauge fixing" condition : minimize: $\mathrm{tr}\,(A_0)^2$ w.r.t. Lorentz tr.

$$\begin{pmatrix} A_0' \\ A_j' \end{pmatrix} = \begin{pmatrix} \cosh\sigma & \sinh\sigma \\ \sinh\sigma & \cosh\sigma \end{pmatrix} \begin{pmatrix} A_0 \\ A_j \end{pmatrix}$$

$\mathrm{tr}\,(A_0 A_j) = 0$ for all j

$$Z = \int dA\, e^{i(S_\mathrm{b}+S_\mathrm{m})} \Delta_\mathrm{FP}[A] \prod_{j=1}^{d} \delta(\mathrm{tr}\,(A_0 A_j))$$

$$\Delta_\mathrm{FP}[A] = \det\Omega \ , \quad \Omega_{ij} = \mathrm{tr}\,(A_0)^2 \delta_{ij} + \mathrm{tr}\,(A_i A_j)$$

- We still have to take care of the oscillating integral by introducing the convergence factor:

$$S_\mathrm{m}^{(\varepsilon)} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon}\,\mathrm{tr}(A_0)^2 - e^{-i\varepsilon}\,\mathrm{tr}(A_i)^2 \right\}$$

Unlike the gauge-unfixed model,
the partition function is finite in the $\varepsilon \to 0$ limit.

2. Classical solutions in N=2 bosonic model

Asano, JN, Piensuk, Yamamori, in preparation

classical solutions for the N=2 case

classical EOM : $[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0$

For N=2, we can obtain all the <u>real</u> solutions up to $\mathrm{SO}(9,1) \times \mathrm{SU}(2)$ sym.

$\gamma < 0$	$A_\mu = 0$		
$\gamma > 0$	$A_\mu = 0$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{8}}\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{4}}\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$

(trivial solution) (Pauli solution) (squashed Pauli solution)

remaining symmetries

$\mathrm{SO}(9,1) \times \mathrm{SU}(2)$ unbroken

diagonal subgroup of $\mathrm{SO}(3) \times \mathrm{SU}(2)$

diagonal subgroup of $\mathrm{SO}(2) \times \mathrm{U}(1)$

Nontrivial real solutions exist only for $\gamma > 0$.

comments on complex solutions

- real solutions are exhausted (up to symmetries) by

$\gamma < 0$	$A_\mu = 0$
$\gamma > 0$	$A_\mu = 0$ $A_\mu = \begin{cases} \sqrt{\frac{\gamma}{8}}\,\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$ $A_\mu = \begin{cases} \sqrt{\frac{\gamma}{4}}\,\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$

(trivial solution) (Pauli solution) (squashed Pauli solution)

- In fact, there are many complex solutions.

e.g.) $\gamma < 0$

$$A_\mu = \begin{cases} i\sqrt{\frac{|\gamma|}{8}}\,\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases} \qquad A_\mu = \begin{cases} i\sqrt{\frac{|\gamma|}{4}}\,\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

$\gamma > 0$

$$\begin{cases} A_0 &= i\sqrt{\frac{\gamma}{8}}\,\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{8}}\,\sigma_2 \\ A_2 &= \sqrt{\frac{\gamma}{8}}\,\sigma_3 \\ A_i &= 0 \quad \text{for } i \geq 3 \end{cases} \qquad \begin{cases} A_0 &= i\sqrt{\frac{\gamma}{4}}\,\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{4}}\,\sigma_2 \\ A_i &= 0 \quad \text{for } i \geq 2 \end{cases}$$

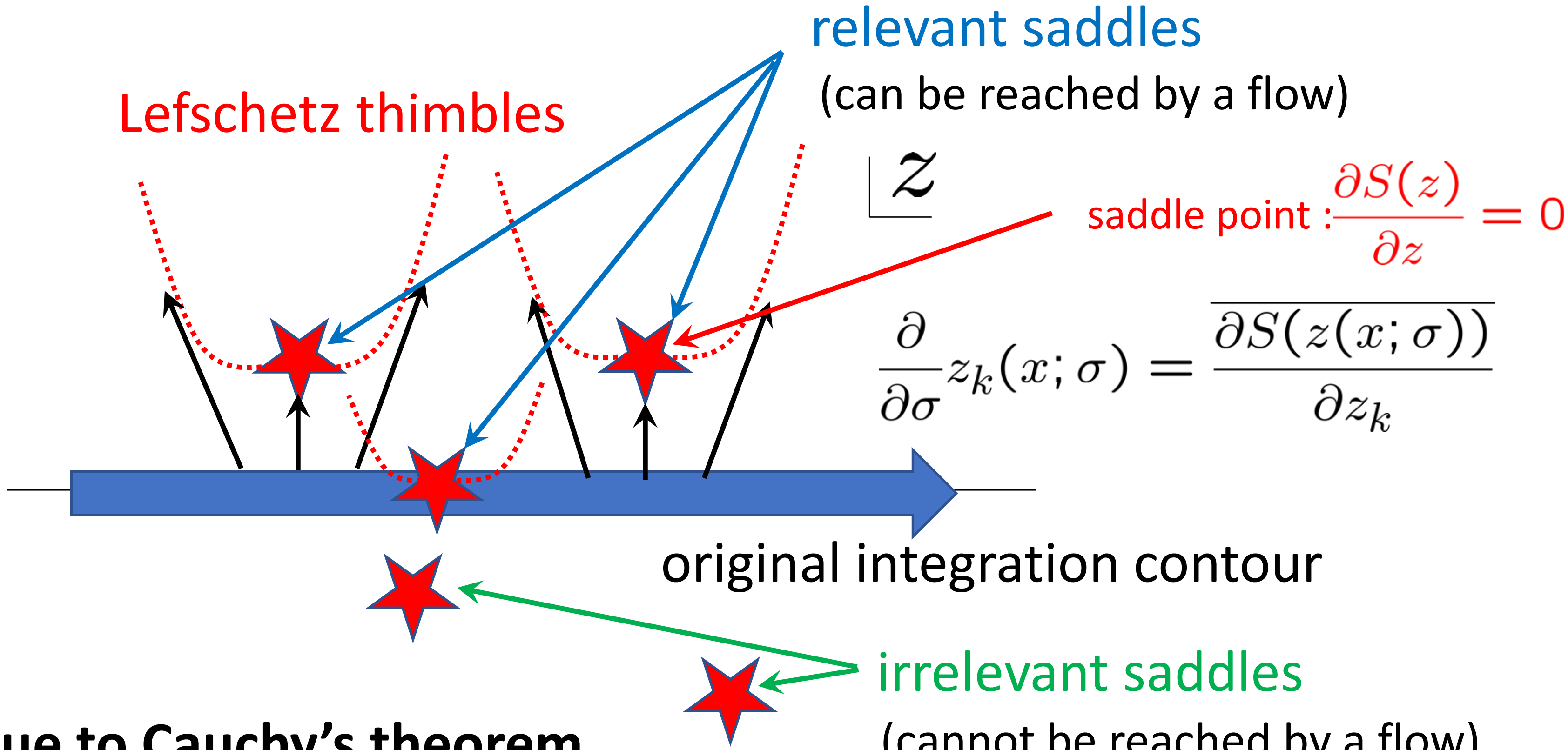
These are all **irrelevant** from the viewpoint of the Picard-Lefschetz theory.

thimble

Picard-Lefschetz theory

(multi-dimensional version of the steepest descent method)

$$Z = \int_{\mathbb{R}^N} dx\, e^{-S(x)} \qquad S(x) \in \mathbb{C}$$ (oscillating integral)



relevant saddles

(can be reached by a flow)

Lefschetz thimbles

z

saddle point : $\frac{\partial S(z)}{\partial z} = 0$

$$\frac{\partial}{\partial \sigma} z_k(x;\sigma) = \overline{\frac{\partial S(z(x;\sigma))}{\partial z_k}}$$

original integration contour

irrelevant saddles

(can<u>not</u> be reached by a flow)

Due to Cauchy's theorem

An oscillating integral can be evaluated by summing over all the thimbles associated with the relevant saddle points.

an important property of the flow

Note that $S(x)$ is complex!

$$
\begin{aligned}
\frac{d}{d\sigma} S(z(x;\sigma)) &= \frac{\partial S(z(x;\sigma))}{\partial z_k} \frac{\partial z_k(x;\sigma)}{\partial \sigma} \\
&= \frac{\partial S(z(x;\sigma))}{\partial z_k} \overline{\frac{\partial S(z(x;\sigma))}{\partial z_k}} \\
&= \left| \frac{\partial S(z(x;\sigma))}{\partial z_k} \right|^2
\end{aligned}
$$

real positive !

Real part of the action increases along the flow,
while the imaginary part is kept constant.

complex saddles are irrelevant in the bosonic model

$$Z = \int dA\, e^{-S[A]} \qquad\qquad S[A] = -i(A^4 + \gamma A^2)$$

$\mathrm{Re}\, S[A] = 0$ for real configurations

$\mathrm{Re}\, S[A] > 0$ required for complex saddles to be relevant

- complex solutions $\qquad \mathrm{Re}\, S[A] = 0$

e.g.) $\gamma < 0$
$$A_\mu = \begin{cases} i\sqrt{\frac{|\gamma|}{8}}\,\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases} \qquad A_\mu = \begin{cases} i\sqrt{\frac{|\gamma|}{4}}\,\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

$\gamma > 0$
$$\begin{cases} A_0 &= i\sqrt{\frac{\gamma}{8}}\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{8}}\sigma_2 \\ A_2 &= \sqrt{\frac{\gamma}{8}}\sigma_3 \\ A_i &= 0 \quad \text{for } i \geq 3 \end{cases} \qquad \begin{cases} A_0 &= i\sqrt{\frac{\gamma}{4}}\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{4}}\sigma_2 \\ A_i &= 0 \quad \text{for } i \geq 2 \end{cases}$$

These are all **irrelevant** from the viewpoint of the Picard-Lefschetz theory.

3. 1/D expansion and MC studies of the cutoff model

Asano, JN, Piensuk, Yamamori, in preparation

1/D expansion

Used in the Euclidean model without the mass term

Hotta-J.N.-Tsuchiya ('98)

$$A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a t^a \qquad h_{ab} \sim A_\mu^a A^{\mu b}$$

For the moment, we omit the convergence factors.

$$\begin{aligned} Z &= \int dA\, e^{i(A^4+\gamma A^2)} \\ &= \int dh \int dA\, e^{i(h^2+hA^2+\gamma A^2)} \\ &= \int dh\, e^{ih^2-\frac{D}{2}\log\det K} \\ &= \int d\tilde{h}\, e^{-DS_{\text{eff}}[\tilde{h}]} \end{aligned}$$

$$\boxed{\begin{aligned} \tilde{h}_{ab} &= \frac{1}{\sqrt{D}} h_{ab} \\ \tilde{\gamma} &= \frac{1}{\sqrt{D}} \gamma \end{aligned}}$$

D appears here only as a parameter.

At large D with fixed $\tilde{\gamma}$,

$$\frac{\partial S_{\text{eff}}[\tilde{h}]}{\partial \tilde{h}_\mu} = 0 \quad \Longrightarrow \quad \tilde{h} + iK[\tilde{h}]^{-1} = 0$$

Large D saddles for N=2 bosonic model

Large D SPE : $\tilde{h} + iK[\tilde{h}]^{-1} = 0$

For N=2, we can obtain all the relevant saddle points up to symmetries.

$\gamma < 0$	$\tilde{h} = v^{(-)}\mathbf{1}$		
$\gamma > 0$	$\tilde{h} = v^{(-)}\mathbf{1}$	$\tilde{h} = v^{(+)}\mathbf{1}$	$\tilde{h} = \tilde{\gamma}\,\mathrm{diag}\left(1, 1, \frac{i}{\tilde{\gamma}^2}\right)$

$$v^{(\pm)} \equiv \frac{1}{4}\tilde{\gamma}\left(1 \pm \sqrt{1 + \frac{8i}{\tilde{\gamma}^2}}\right)$$

These are complex saddles, but not necessarily irrelevant.

remaining symmetries	SU(2)	SU(2)	U(1)
identification	⇕	⇕	⇕
	trivial solution	Pauli solution	squashed Pauli solution
	$\mathrm{SO}(9,1) \times \mathrm{SU}(2)$ unbroken	diagonal subgroup of $\mathrm{SO}(3) \times \mathrm{SU}(2)$	diagonal subgroup of $\mathrm{SO}(2) \times \mathrm{U}(1)$

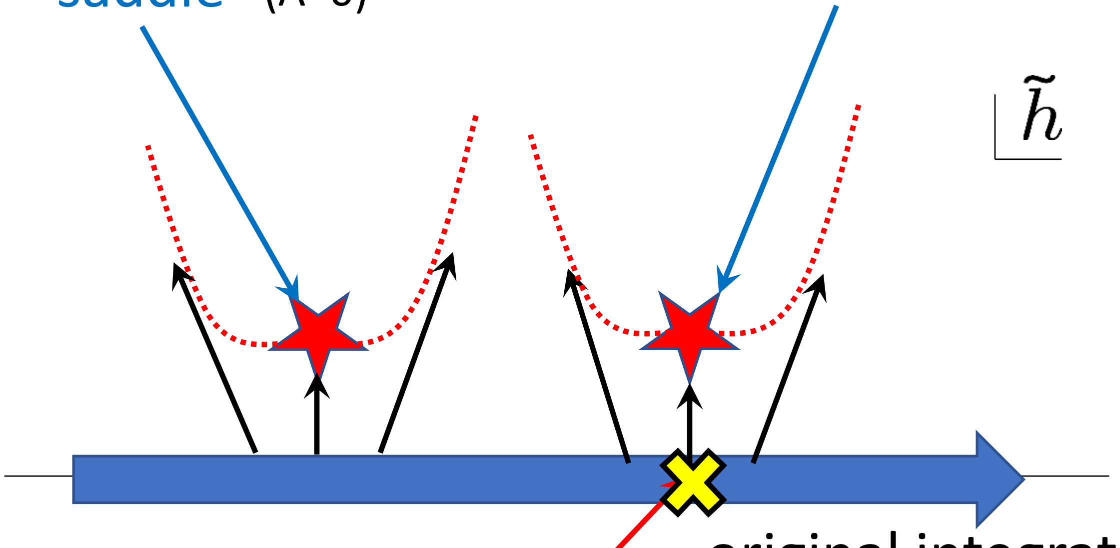
Singularity on the real axis

$\tilde{h} = v^{(-)}\mathbf{1}$

v$^{(-)}$ saddle (A=0)

$\tilde{h} = v^{(+)}\mathbf{1}$

v$^{(+)}$ saddle (Pauli)



$$v^{(\pm)} \equiv \frac{1}{4}\tilde{\gamma}\left(1 \pm \sqrt{1+\frac{8i}{\tilde{\gamma}^2}}\right)$$

$\tilde{\gamma} > 0$

original integration contour

$\tilde{h}_{\text{sing}} = \frac{\tilde{\gamma}}{2}\mathbf{1}$

singularity

$\det K[\tilde{h}_{\text{sing}}] = 0$

This simply reflects the fact that the partition function is not well defined as it is.

Also true for the SO(D) invariant case !

The case of SO(D) symmetric model obtained by replacing $A_0 = iA_D$

$$Z^{(+)} \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} \left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D}$$

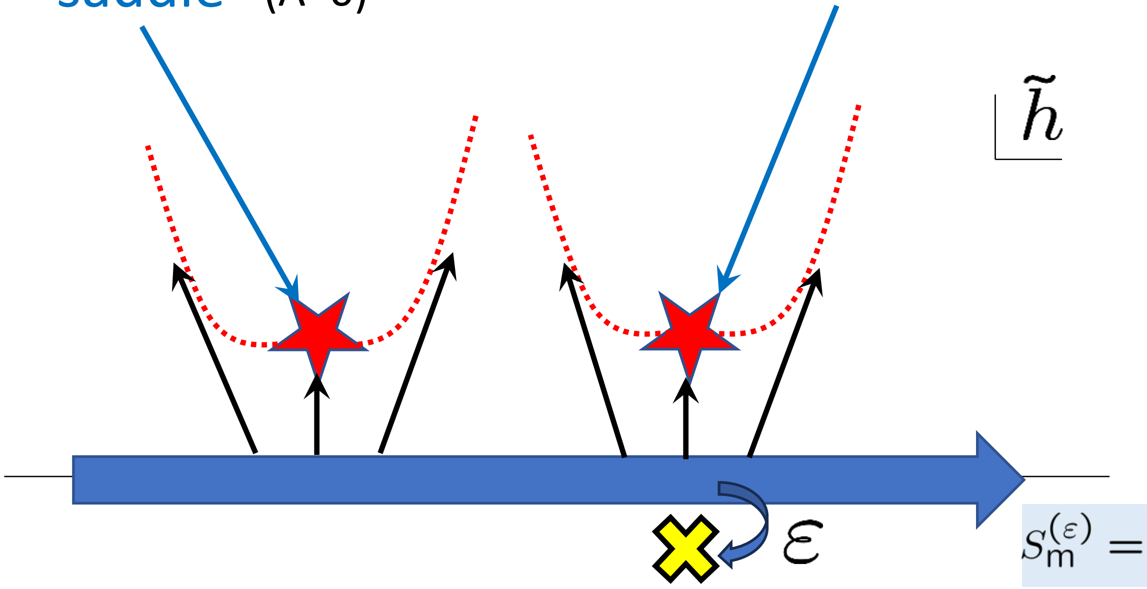
(large D)

$\tilde{h} = v^{(-)} \mathbf{1}$

$v^{(-)}$ saddle (A=0)

$\tilde{h} = v^{(+)} \mathbf{1}$

$v^{(+)}$ saddle (Pauli)



$$v^{(\pm)} \equiv \frac{1}{4}\tilde{\gamma}\left(1 \pm \sqrt{1 + \frac{8i}{\tilde{\gamma}^2}}\right)$$

$$\tilde{\gamma} > 0$$

$$S_{\rm m}^{(\varepsilon)} = -\frac{1}{2} N\gamma\, e^{-i\varepsilon} \left\{ \mathrm{tr}(A_D)^2 + \mathrm{tr}(A_i)^2 \right\}$$

The $v^{(+)}$ saddle becomes relevant and

the associated partition function becomes finite in the $\varepsilon \to 0$ limit.

For $\tilde{\gamma} < 0$, the $v^{(+)}$ saddle becomes irrelevant

since the singularity is shifted in the opposite direction.

consistent with the existence of the Pauli solution only for $\tilde{\gamma} > 0$.

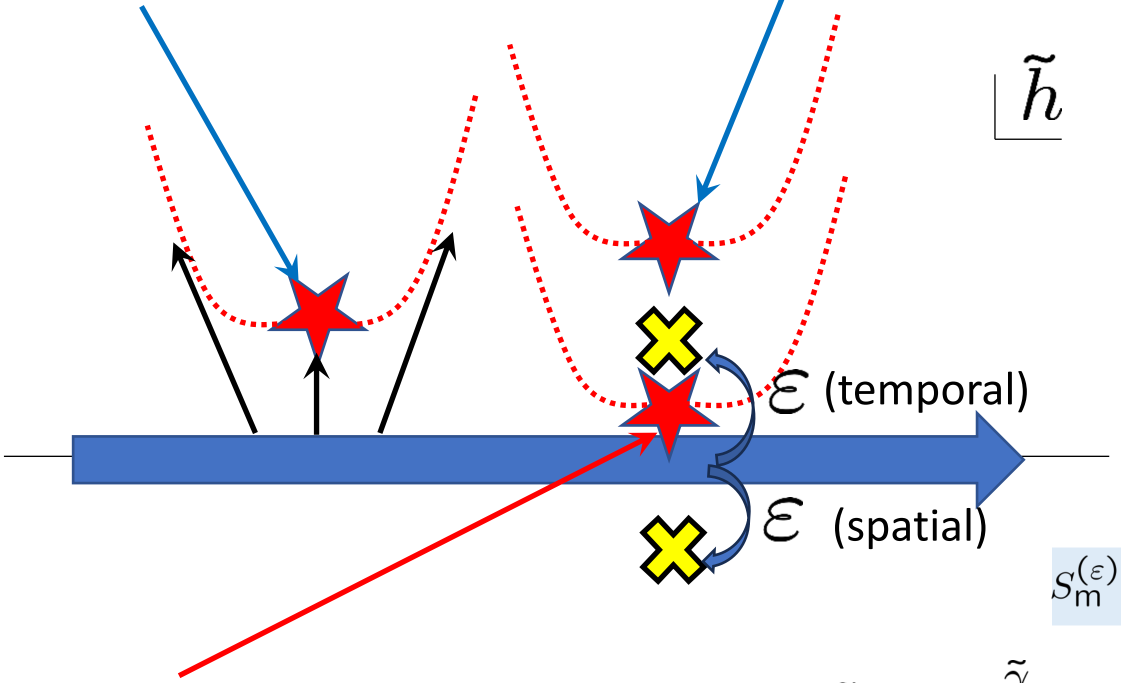
The case of Lorentz symmetric model

$\tilde{h} = v^{(-)}\mathbf{1}$

v$^{(-)}$ saddle (A=0)

$\tilde{h} = v^{(+)}\mathbf{1}$

v$^{(+)}$ saddle becomes irrelevant!



$$v^{(\pm)} \equiv \frac{1}{4}\tilde{\gamma}\left(1 \pm \sqrt{1+\frac{8i}{\tilde{\gamma}^2}}\right)$$

$$\tilde{\gamma} > 0$$

Convergence factor acts on space and time differently.

$$S_{\mathrm{m}}^{(\varepsilon)} = \frac{1}{2}N\gamma\left\{e^{i\varepsilon}\mathrm{tr}(A_0)^2 - e^{-i\varepsilon}\mathrm{tr}(A_i)^2\right\}$$

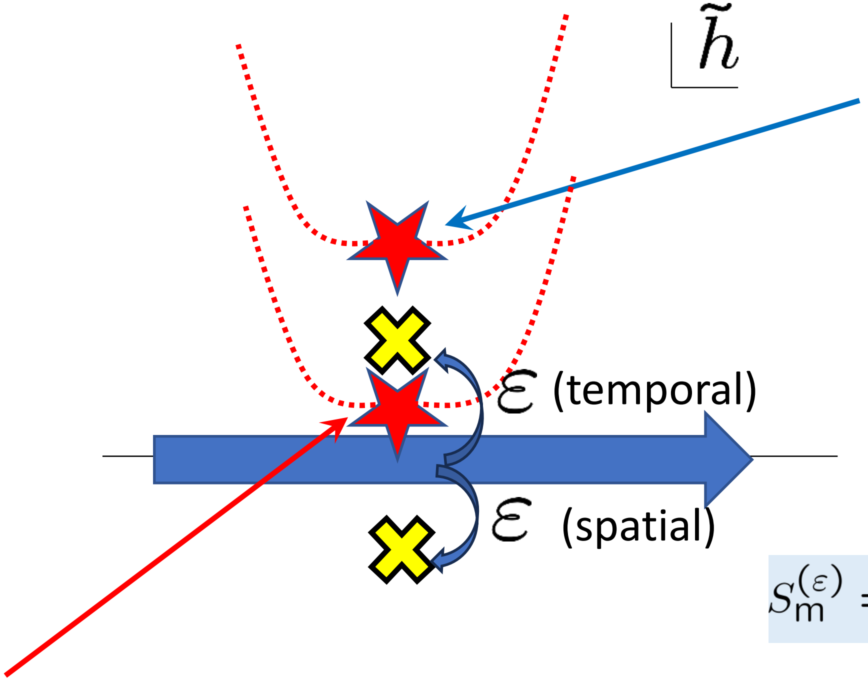
new saddle point appears near $\tilde{h}_{\mathrm{sing}} = \frac{\tilde{\gamma}}{2}\mathbf{1}$

$$Z_{\mathrm{Pauli}} \sim \varepsilon^{-\frac{3}{2}D}(\times\varepsilon^6) \sim \varepsilon^{-(\frac{3}{2}D-6)}$$

diverges as $\varepsilon \to 0$

The new (relevant) saddle point appears, and the partition function diverges!

Situation with the squashed Pauli



$$\tilde{h} \;=\; \tilde{\gamma}\,\text{diag}\left(1, 1, \frac{i}{\tilde{\gamma}^2}\right)$$

This becomes irrelevant!

$\tilde{\gamma} > 0$

Convergence factor acts on space and time differently.

$$S_{\rm m}^{(\varepsilon)} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon}\,\text{tr}(A_0)^2 - e^{-i\varepsilon}\,\text{tr}(A_i)^2 \right\}$$

new saddle point appears near $\tilde{h}_{\text{sing}} \;=\; c\,\mathbf{1} + (2c - \tilde{\gamma})\,\text{diag}\left(1, 1, \frac{i}{\tilde{\gamma}^2}\right)$

$$c = \frac{\tilde{\gamma}}{12}\left(7 + \sqrt{1 - \frac{8i}{\tilde{\gamma}}}\right)$$

$$Z_{\text{s}-\text{Pauli}} \sim \varepsilon^{-D}(\times \varepsilon^3) \sim \varepsilon^{-(D-3)}$$

diverges as $\varepsilon \to 0$

The new (relevant) saddle point appears, and the partition function diverges!

Physical meaning of the divergence

$$Z \sim \varepsilon^{-p}$$

Note: This does not mean that the model is ill defined.
E.g., the expectation value $\langle \mathrm{tr}\,(A_\mu A^\mu) \rangle$ is finite.

Pauli $p \sim \frac{3}{2}D - 6$

squashed Pauli $p \sim D - 3$

Partition function divegerges faster for Pauli for $D \gtrsim 6$

Pauli has 3 nonvanishing internal d.o.f., while squashed Pauli has only 2.

This implies that Pauli thimble dominates in the cutoff model at $\gamma > 0$ for $D \gtrsim 6$.

The diverging observables :

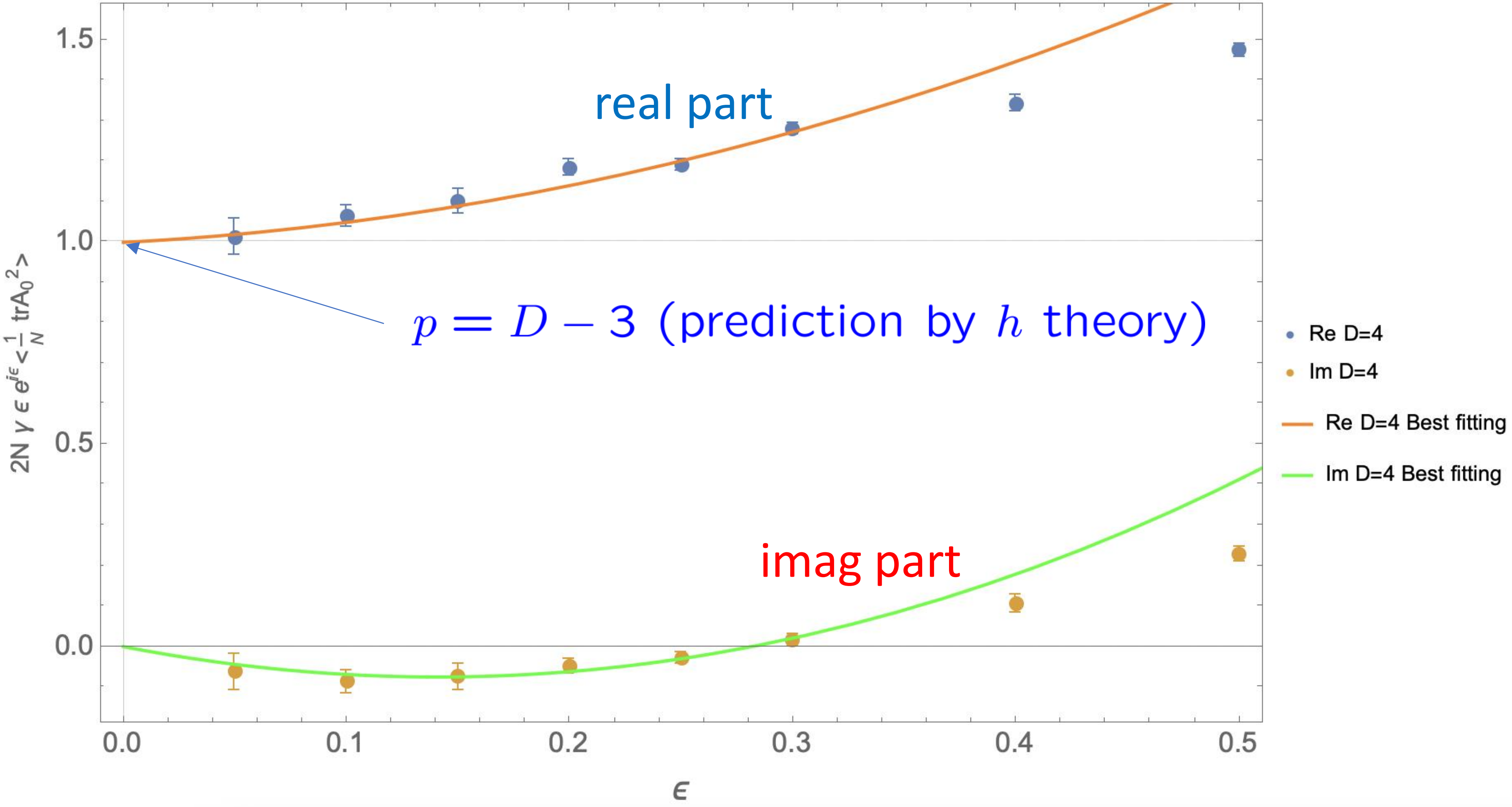
$$\left\langle \frac{1}{N} \mathrm{tr}\,(A_0)^2 \right\rangle \sim -\frac{1}{2N\gamma\, e^{i\varepsilon}} \frac{\partial}{\partial \varepsilon} \log Z \sim \frac{p}{2N\gamma\, e^{i\varepsilon}\, \varepsilon}$$

Boosted configurations dominate the partition function.
The cutoff artifact may well remain in the $\varepsilon \to 0$ limit.

Diverging

$$\left\langle \frac{1}{N}\mathrm{tr}\,(A_0)^2 \right\rangle \sim \frac{p}{2N\gamma\, e^{i\varepsilon}\,\varepsilon}$$

$\left\langle \frac{1}{N}\mathrm{tr}\,(A_0)^2 \right\rangle \times 2N\gamma\, e^{i\varepsilon}\varepsilon$ $D = 4,$ squashed Pauli $\gamma = 1.5$



real part

$p = D - 3$ (prediction by h theory)

imag part

Consistent with $Z \sim \varepsilon^{-(D-3)}$

Classicalization for Pauli solution

- The new saddle point approaches $\tilde{h}_{\text{sing}} = \frac{\tilde{\gamma}}{2}\mathbf{1}$

⬇

$$\frac{1}{\sqrt{D}}\frac{1}{N}\langle \text{tr}\, A_\mu A^\mu \rangle = \frac{1}{4}\langle \text{tr}\, \tilde{h} \rangle \sim \frac{3}{8}\tilde{\gamma}$$

- Fluctuations around the saddle point are suppressed at large D.

$$\lim_{D\to\infty}\frac{1}{\sqrt{D}}\frac{1}{N}\langle \text{tr}\, A_\mu A^\mu \rangle = \frac{3}{8}\tilde{\gamma}$$

(classical result)

$$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{8}}\,\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Classicalization for Pauli occurs at $D = \infty$.

Hessian analysis around boosted Pauli

$$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{8}}\,\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

boost in
1-direction

$$\begin{cases} A_0 &= \sqrt{\frac{\gamma}{8}}\,\sinh\sigma\,\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{8}}\,\cosh\sigma\,\sigma_1 \\ A_2 &= \sqrt{\frac{\gamma}{8}}\,\sigma_2 \\ A_3 &= \sqrt{\frac{\gamma}{8}}\,\sigma_3 \\ A_i &= 0 \quad \text{for } i \geq 4 \end{cases}$$

$$\delta S = \delta A_\mu^a\, H_{\mu\nu}^{ab}(A)\, \delta A_\nu^b \qquad H_{\mu\nu}^{ab} = \frac{\partial^2 S(A)}{\partial A_\mu^a \partial A_\nu^b} \quad \text{(Hessian)}$$

cutoff surface: $\mathrm{tr}\,(A_0)^2 + \mathrm{tr}\,(A_i)^2 = \Lambda$

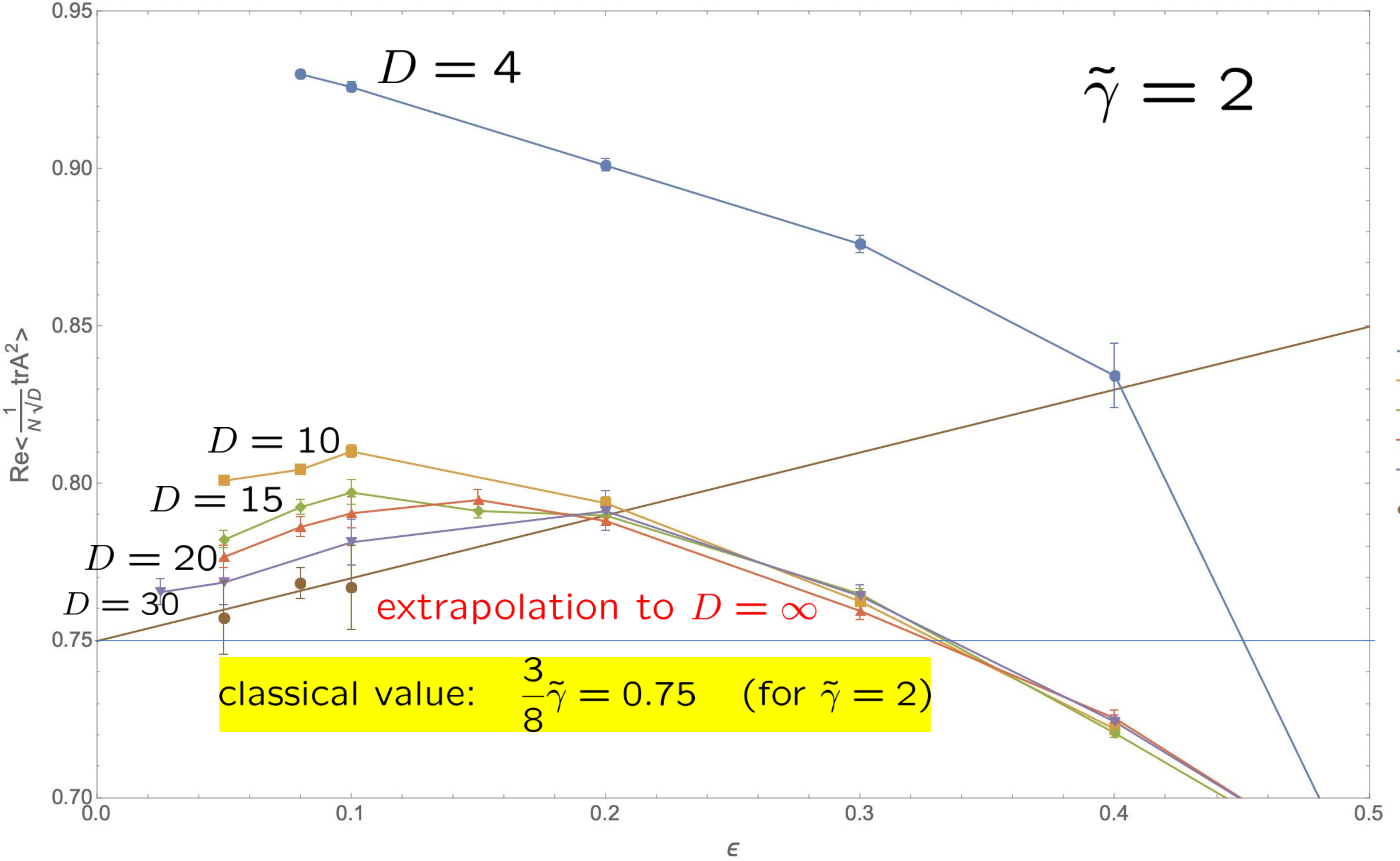
allowed fluctuations: $\mathrm{tr}\,(A_0 \delta A_0) + \mathrm{tr}\,(A_i \delta A_i) = 0$

Eigenvalues of H in the $(3D-1)$-dimensional subspace

2	finite	(contributes to quantum corrections)
4	divergent	(suppressed in the $\varepsilon \to 0$ limit)
$(3D-7)$	zeroes	(corresponding to broken symmetries)

Classicalization occurs in the large D limit.

Classicalization for Pauli at $D = \infty$



Hessian analysis around boosted squashed Pauli

$$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{4}}\,\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

boost in 1-direction

$$\begin{cases} A_0 &= \sqrt{\frac{\gamma}{4}}\sinh\sigma\,\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{4}}\cosh\sigma\,\sigma_1 \\ A_2 &= \sqrt{\frac{\gamma}{4}}\,\sigma_2 \\ A_i &= 0 \quad \text{for } i \geq 3 \end{cases}$$

$$\delta S = \delta A_\mu^a\, H_{\mu\nu}^{ab}(A)\,\delta A_\nu^b \qquad H_{\mu\nu}^{ab} = \frac{\partial^2 S(A)}{\partial A_\mu^a \partial A_\nu^b}$$

(Hessian)

cutoff surface: $\mathrm{tr}\,(A_0)^2 + \mathrm{tr}\,(A_i)^2 = \Lambda$

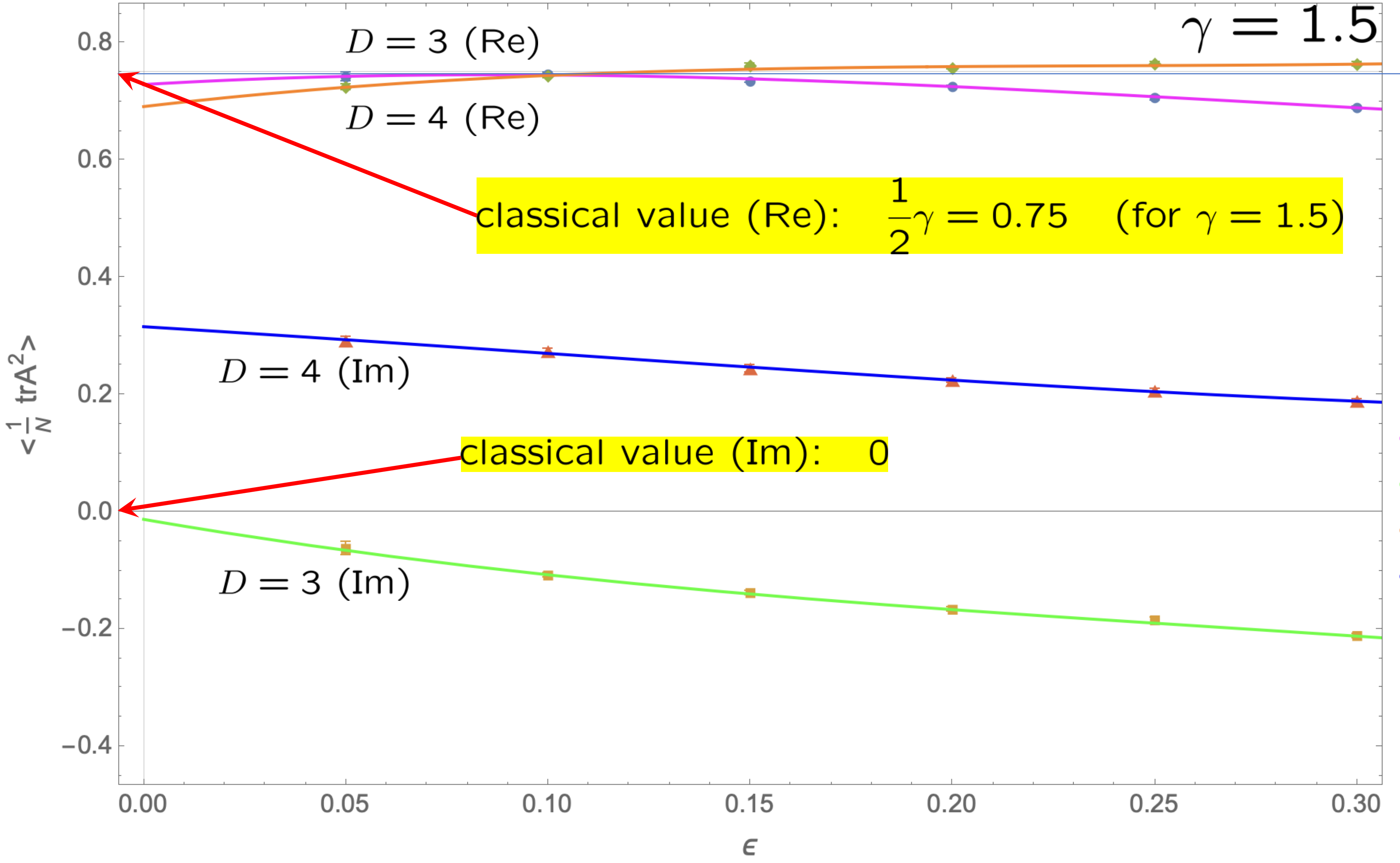
allowed fluctuations: $\mathrm{tr}\,(A_0\delta A_0) + \mathrm{tr}\,(A_i\delta A_i) = 0$

Eigenvalues of H in the $(3D-1)$-dimensional subspace

$(D-3)$	finite	(contributes to quantum corrections)
4	divergent	(suppressed in the $\varepsilon \to 0$ limit)
$(2D-2)$	zeroes	(corresponding to broken symmetries)

Classicalization for squashed pauli occurs only at $D = 3$.

Classicalization for squashed Pauli at $D = 3$



classical value (Re): $\frac{1}{2}\gamma = 0.75$ (for $\gamma = 1.5$)

classical value (Im): 0

Classicalization occurs only at $D = 3$.

4. MC studies of the “gauge-fixed” model

Chou, JN, Tripathi, in preparation

Saddle points in the gauge-fixed model

$$Z = \int dA\, e^{i(S_{\mathrm{b}}+S_{\mathrm{m}})}\, \Delta_{\mathrm{FP}}[A] \boxed{\prod_{j=1}^{d} \delta(\mathrm{tr}\,(A_0 A_j))}$$

$$\Delta_{\mathrm{FP}}[A] = \det \Omega$$

$$\Omega_{ij} = \mathrm{tr}\,(A_0)^2 \delta_{ij} + \mathrm{tr}\,(A_i A_j)$$

This represents the gauge fixing condition : $\mathrm{tr}\,(A_0 A_j) = 0$ for all j

saddle point equation :

$$[A_\nu, [A^\nu, A_\mu]] = \gamma A_\mu + \frac{i}{N} \eta_{\mu\nu} \mathrm{Tr} \left(\Omega^{-1} \frac{\partial \Omega}{\partial A_\nu} \right)$$

Using the SO(d) symmetry, we can impose : $\mathrm{tr}\,(A_i A_j) = 0$ for $i \neq j$

$$[A_\nu, [A^\nu, A_\mu]] = (\gamma + i\kappa_\mu) A_\mu ,$$

The effect of gauge-fixing appears here.

$$\kappa_i = \frac{2}{N\{\mathrm{tr}\,(A_0)^2 + \mathrm{tr}\,(A_i)^2\}} , \quad \kappa_0 = \sum_{i=1}^{d} \kappa_i .$$

κ_μ has to be determined in a self-consistent manner.

The FP determinant induces a mass-like term in the saddle point eq.

The $\gamma \to 0$ limit may be smooth!

Ansatz for the saddle points

$$[A_\nu, [A^\nu, A_\mu]] = (\gamma + i\kappa_\mu) A_\mu \ , \qquad \kappa_i = \frac{2}{N\{\mathrm{tr}\,(A_0)^2 + \mathrm{tr}\,(A_i)^2\}} \ , \qquad \kappa_0 = \sum_{i=1}^{d} \kappa_i \ .$$

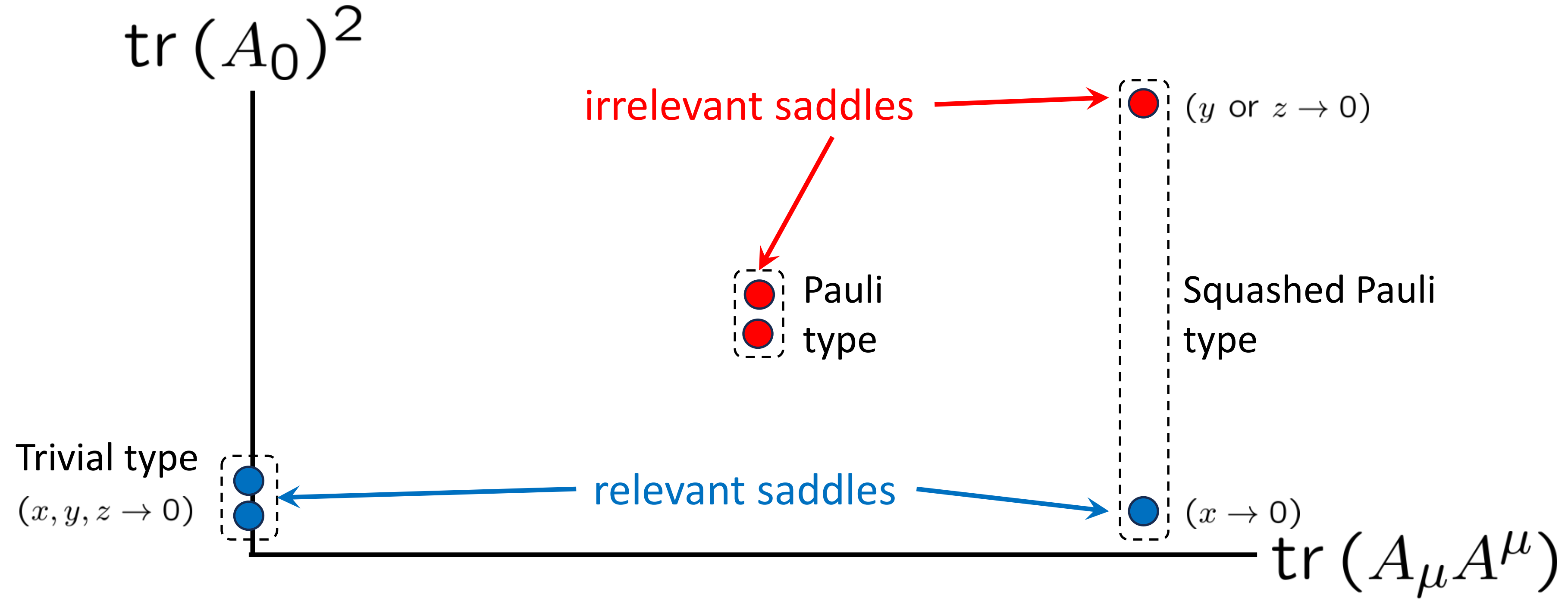
A natural ansatz:
$d \geq 4$

$$A_0 = x\,\sigma_1 \ ,$$
$$A_1 = y\,\sigma_2 \ ,$$
$$A_2 = z\,\sigma_3 \ ,$$
$$A_j = 0 \qquad \text{for } 3 \leq j \leq d$$

$A_0 \neq 0$ is required for finite γ since otherwise $\Delta_{\mathrm{FP}}[A] = 0$.

solutions at $\gamma > 0$

(“type” = $\gamma \to \infty$ behavior)



The behavior of the solutions at $\gamma \to \infty$

For $\gamma \to \infty$, the solutions reduce to those of the gauge-unfixed model

$\gamma > 0$	$A_\mu = 0$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{8}}\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{4}}\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$

(trivial solution) (Pauli solution) (squashed Pauli solution)

Recall, however, that solutions that are obtained by Wick rotation from above are irrelevant from the viewpoint of the Picard-Lefschetz theory.

$$\begin{cases} A_0 &= i\sqrt{\frac{\gamma}{8}}\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{8}}\sigma_2 \\ A_2 &= \sqrt{\frac{\gamma}{8}}\sigma_3 \\ A_i &= 0 \quad \text{for } i \geq 3 \end{cases} \qquad \begin{cases} A_0 &= i\sqrt{\frac{\gamma}{4}}\sigma_1 \\ A_1 &= \sqrt{\frac{\gamma}{4}}\sigma_2 \\ A_i &= 0 \quad \text{for } i \geq 2 \end{cases}$$

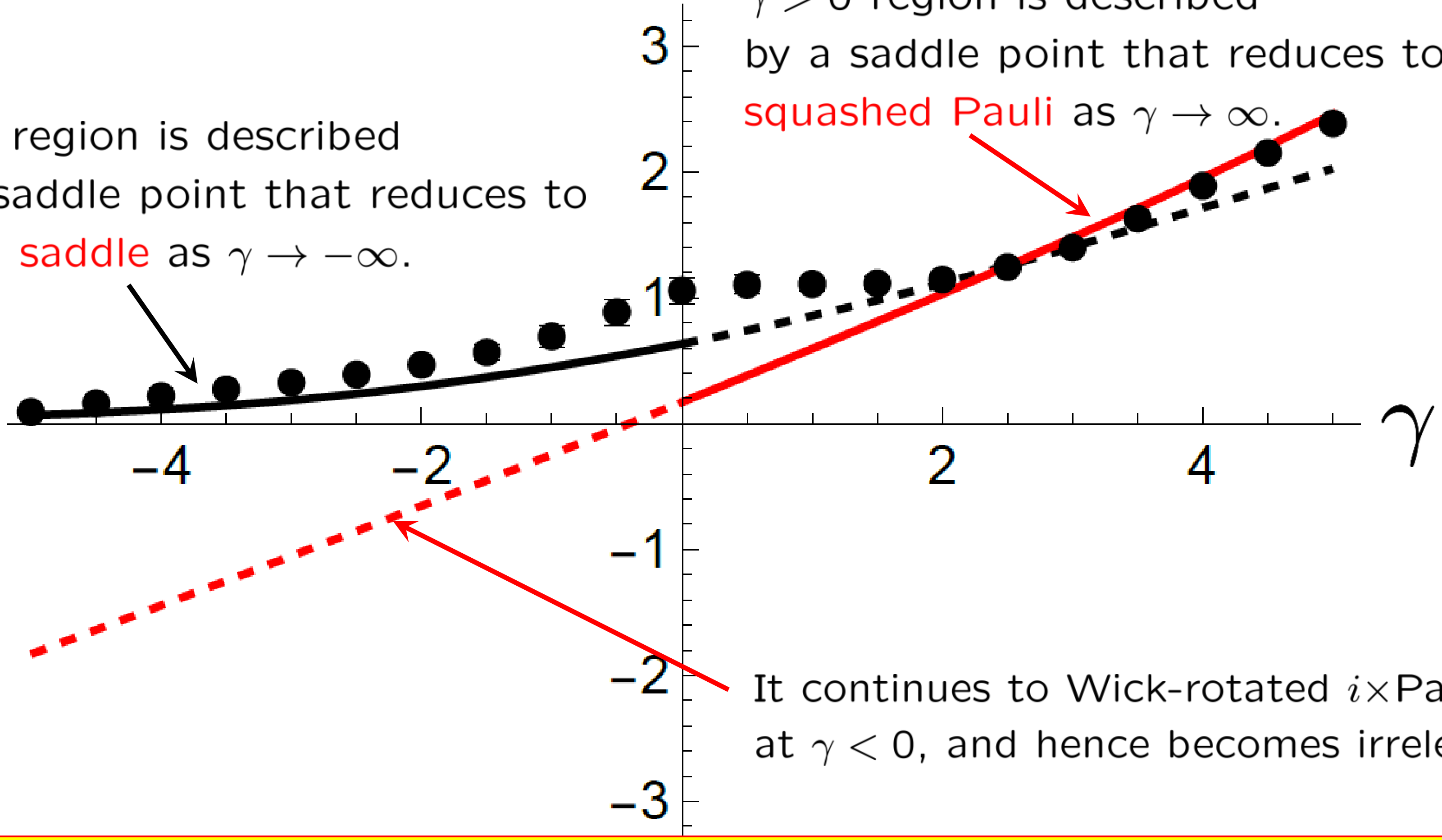
Thus at large γ, relevant saddles should have $A_0 \to 0$.

Pauli-type solution cannot be relevant.

Simulation results for the gauge fixed model (by the generalized Lefschetz thimble method)

$D = 5$

$$\mathrm{Re}\left\langle \frac{1}{N}\mathrm{tr}\left(A_\mu A^\mu\right)\right\rangle$$



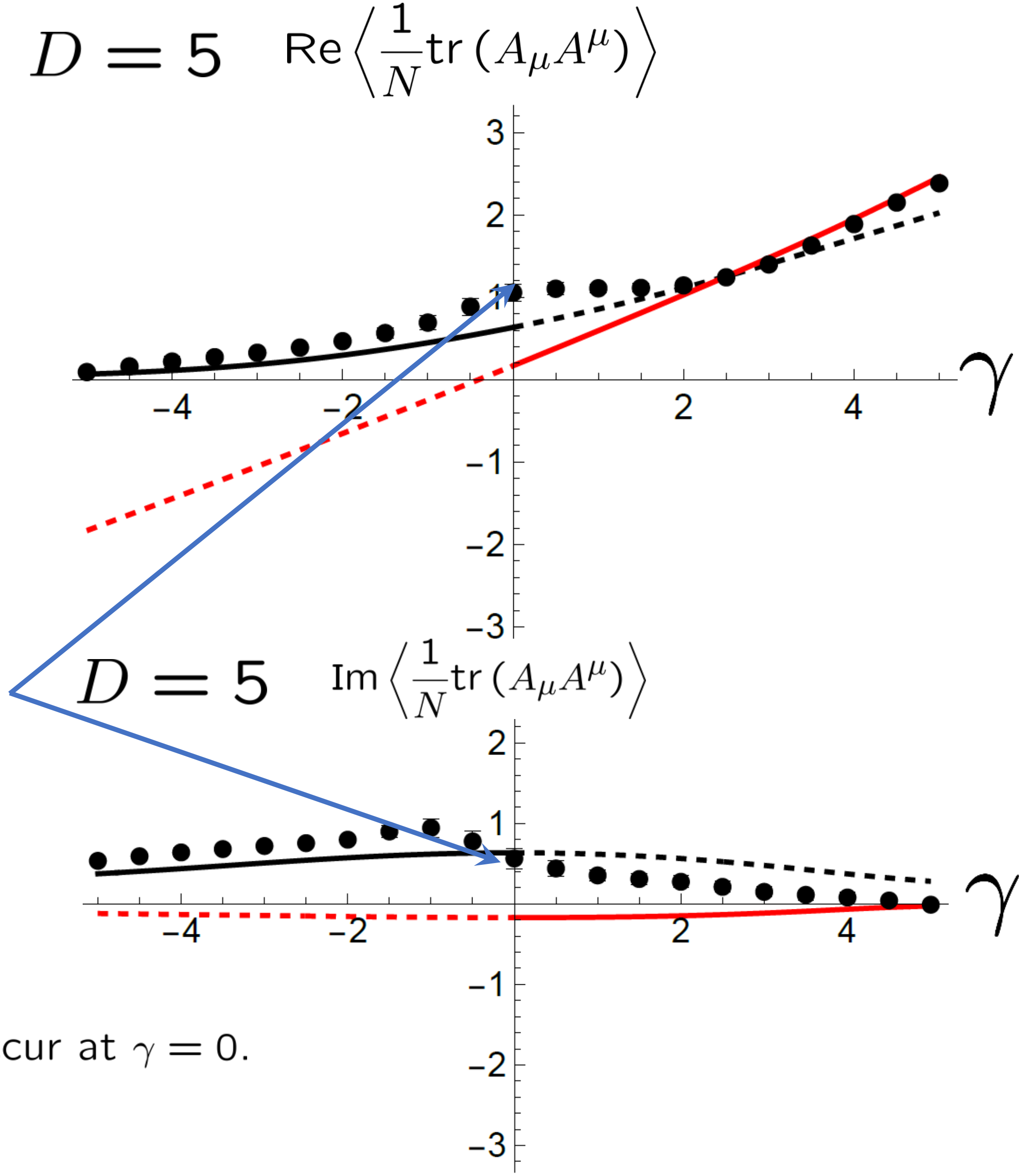
$\gamma > 0$ region is described by a saddle point that reduces to squashed Pauli as $\gamma \to \infty$.

$\gamma < 0$ region is described by a saddle point that reduces to trivial saddle as $\gamma \to -\infty$.

It continues to Wick-rotated $i\times$Pauli at $\gamma < 0$, and hence becomes irrelevant.

Thus, the dominant saddle point for $\gamma > 0$ is different from the cutoff model !

Smooth behavior observed at $\gamma \sim 0$

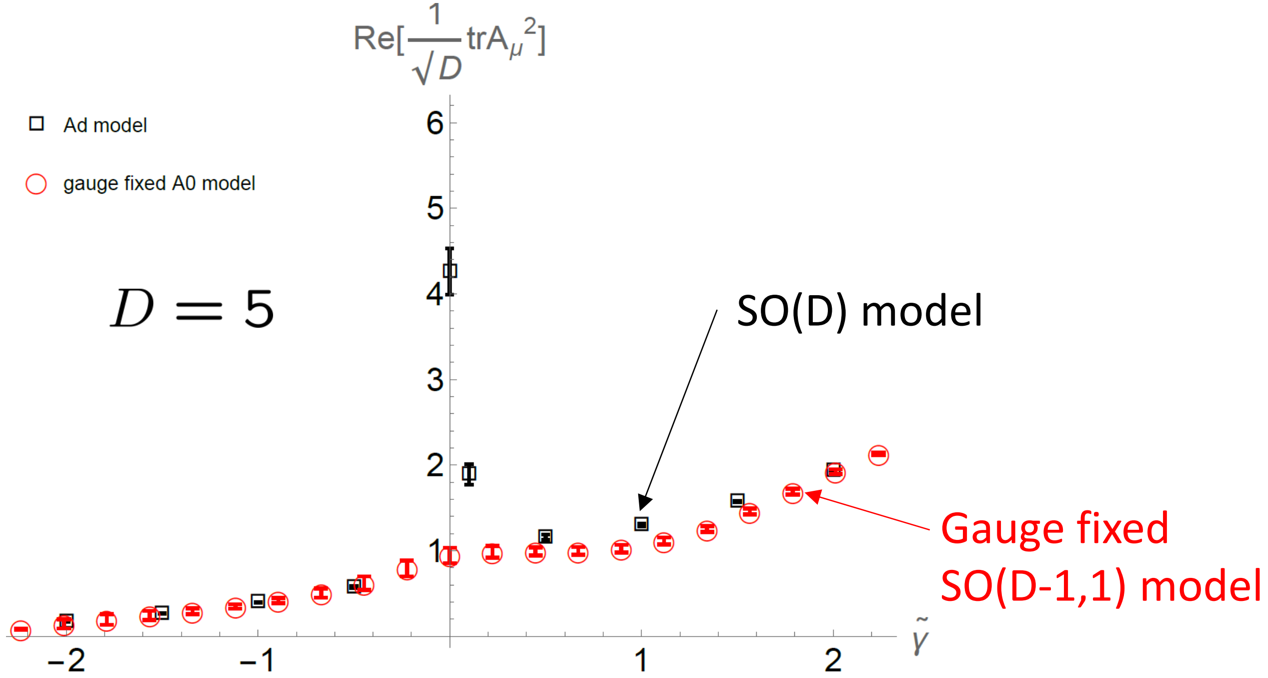


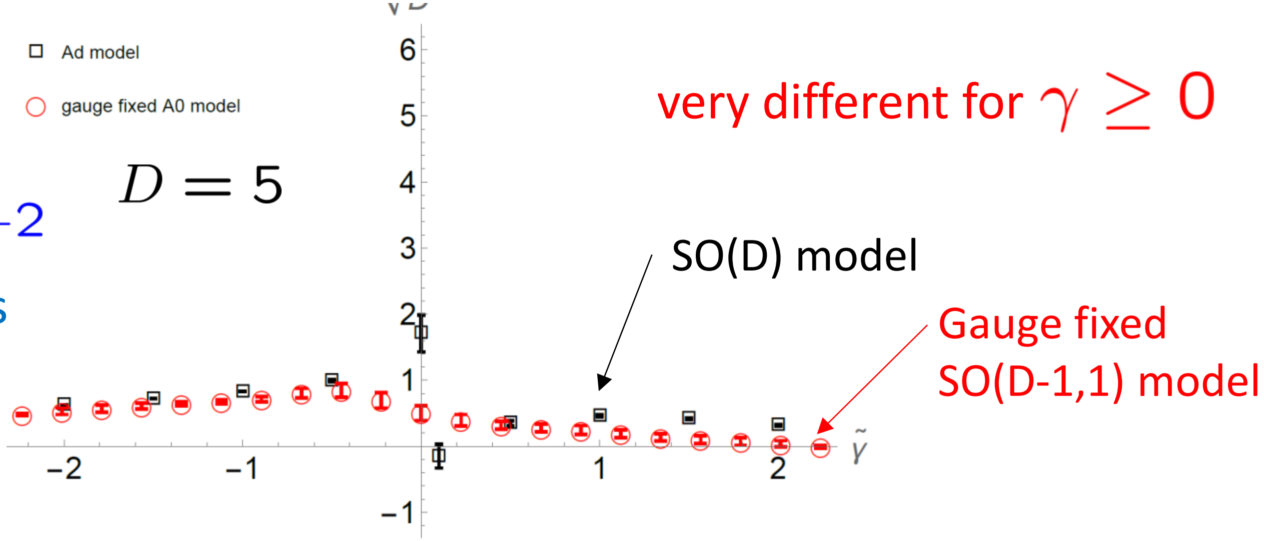
$\gamma \to 0$ limit seems to be smooth at least in this model

But, this might not be the case with SUSY and/or at large N.

Stokes phenomenon seems to occur at $\gamma = 0$.
(Relevant saddle point changes.)

Comparison with the SO(D) symmetric model obtained by replacing $A_0 = iA_D$



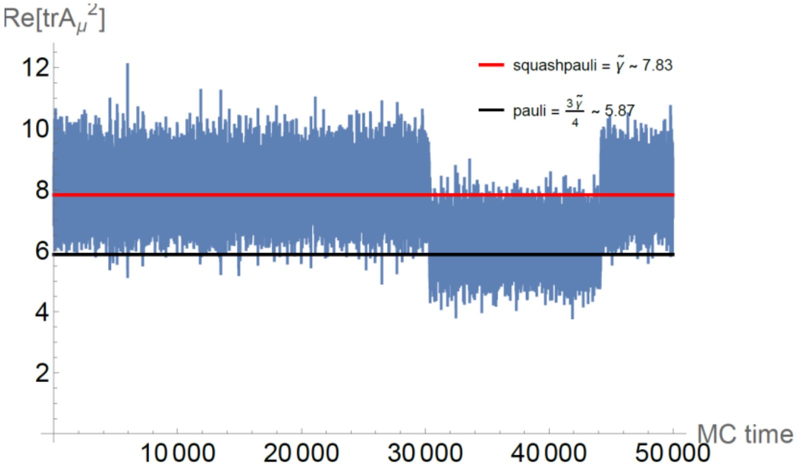


very different for $\gamma \geq 0$

equivalence for $\gamma \lesssim -2$

trivial saddle dominates
in this region

Ocsillating behavior in the SO(D) model at larger $\tilde{\gamma}$



Perturbative calculations around
Pauli and squashed Pauli yield:

$$Z_{\text{Pauli}} \simeq \frac{\pi^{\frac{3(D+1)}{2}}\gamma^{\frac{3D}{2}-6}\boxed{e^{-\frac{3i}{8}\gamma^2}}}{2^{3(D-4)}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(\frac{D-2}{2}\right)}$$

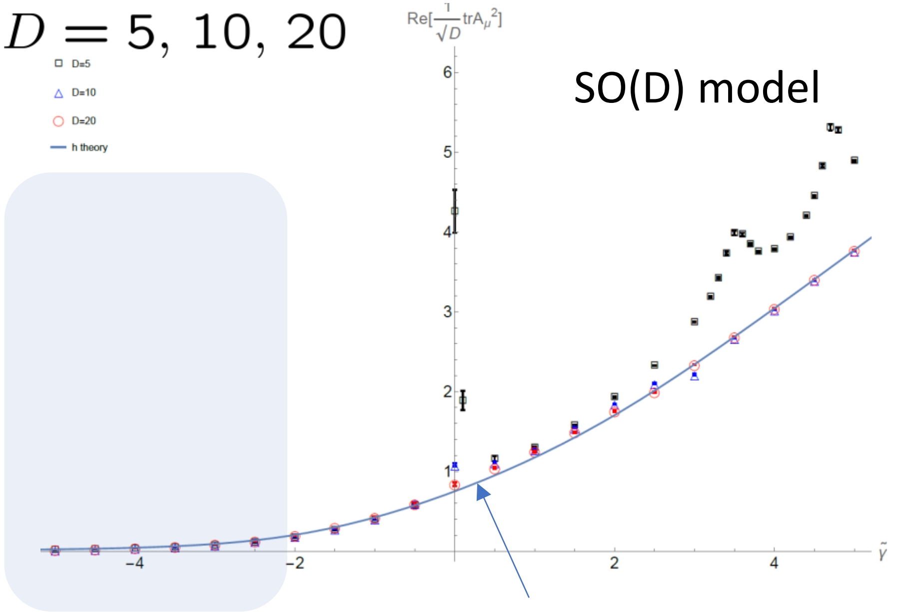
$$Z_{\text{s}-\text{Pauli}} \simeq \frac{\pi^{\frac{3D+2}{2}}\gamma^{\frac{D}{2}-1}\boxed{e^{-\frac{i}{2}\gamma^2}}}{2^{D-\frac{7}{2}}(-i)^{\frac{D-1}{2}}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}$$

Due to the relevative phase,
interference occurs between P and sP.

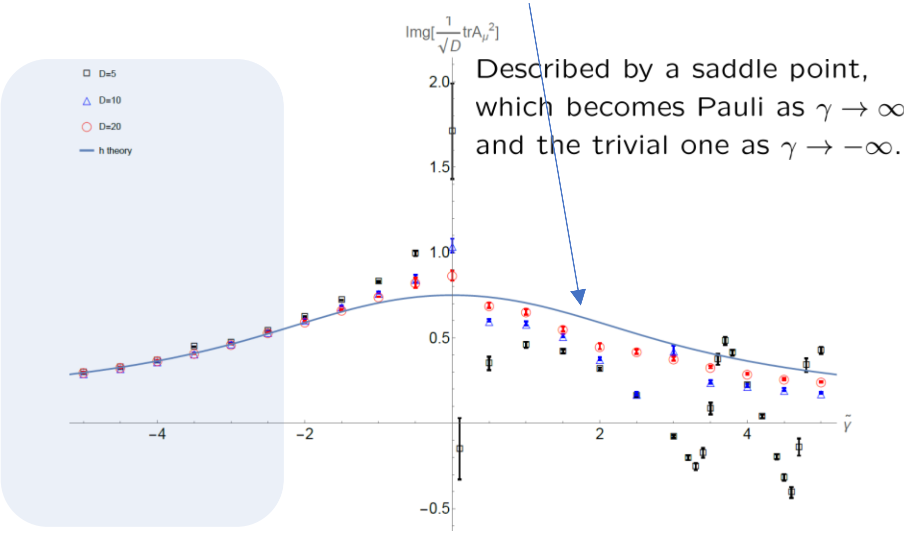
At $D=\infty$, Pauli dominates over s-Pauli.

$D = 5,\ 10,\ 20$

SO(D) model



Prediction at $D=\infty$



Described by a saddle point,
which becomes Pauli as $\gamma \to \infty$
and the trivial one as $\gamma \to -\infty$.

5. Summary and discussions

Summary

- The type IIB matrix model has diverging partition function due to Lorentz symmetry (represented by a noncompact group).

- In the cutoff model, the Pauli solution has more divergent partition function than the squashed Pauli, and hence dominates.

  Pauli has 3 nonvanishing internal d.o.f.,
  while squashed Pauli has only 2.

- The cutoff model suffers from a severe artifact due to Lorentz symmetry breaking. (Classicalization at D=∞ Pauli, D=3 sPauli.)

- We have proposed a new definition of type IIB matrix model without Lorentz symmetry breaking using the gauge fixing. In the gauge-fixed model, Pauli solution cannot appear at large γ .

Gauge fixing is crucial in determining the dominant saddle point.

Future prospects

- What happens in the SUSY case and/or at larger N.

  Does (3+1)-dimensional expanding space-time emerge
  in the $1)\ N \to \infty, 2)\ \gamma \to 0$ limit ?

- SUSY case

  1/D expansion cannot be applied (SUSY cannot be respected),
  but numerical simulation is doable. N=2 case is on-going.

- larger N

  The computational cost of the generalized Lefschetz thimble method
  grows with N as $O(N^6)$. But we may still do N=4,8,16,…

- SUSY and large N

The Pfaffian seems to prefer collapsed configurations, but it becomes **zero**
for configurations with not more than 2 extended directions.

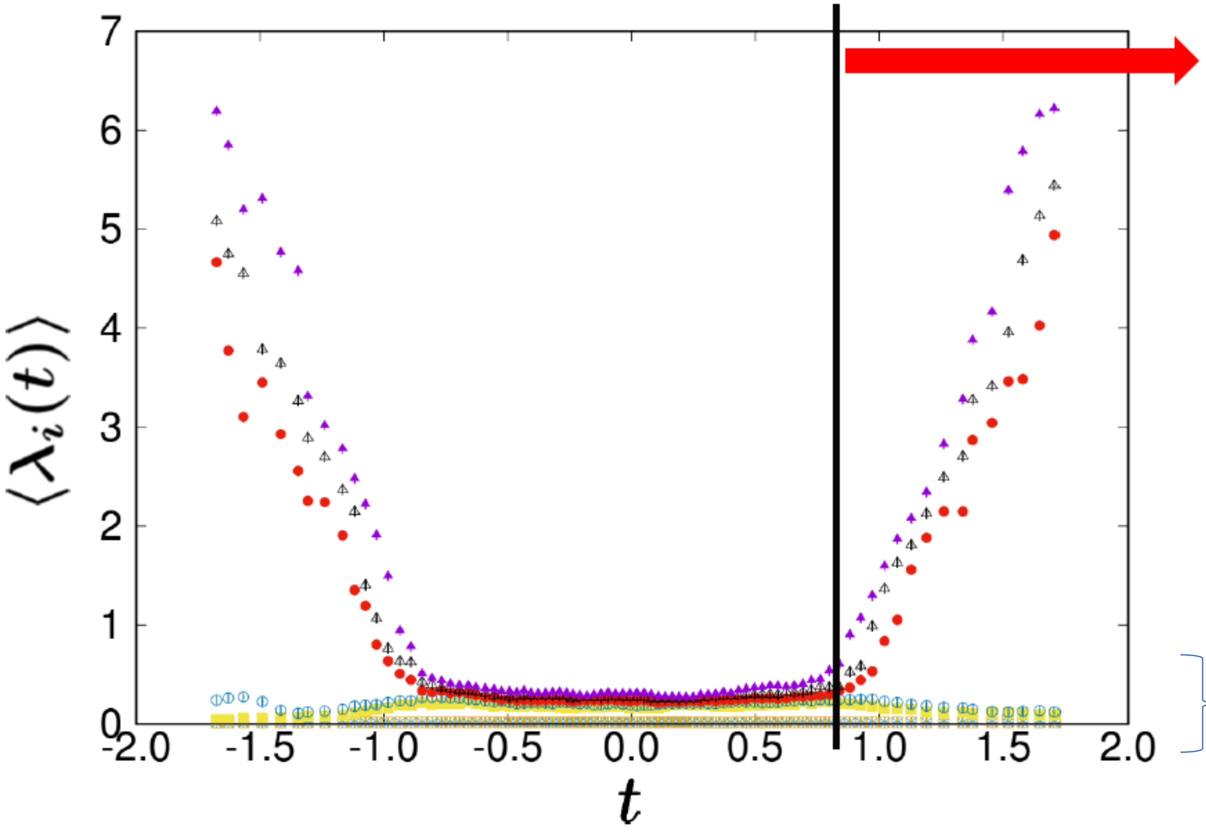
⇨ 3d space ? complex Langevin method (less flexible but cheaper)

Recent results from complex Langevin simulation

(gauge-unfixed model)

Anagnostopoulos, Azuma, Hatakeyama, Hirasawa, JN, Papadoudis, Tsuchiya, in preparation

The extent in 9 directions v.s. time



Emergence of expanding (3+1)D space-time

$$N = 96$$

$$\gamma = 4$$

$$m_{\mathrm{f}} = 3.5$$

$$d = 5\,, \quad \xi = 16$$

5 directions are suppressed by hand for technical reasons.

The 4th direction becomes small at late times spontaneously !