No Smooth Spacetime in Lorentzian Quantum Cosmology

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[Phys.Rev.D 110 (2024) 2, 023503, Phys.Rev.D 107 (2023) 4, 043511, 2402.09981]

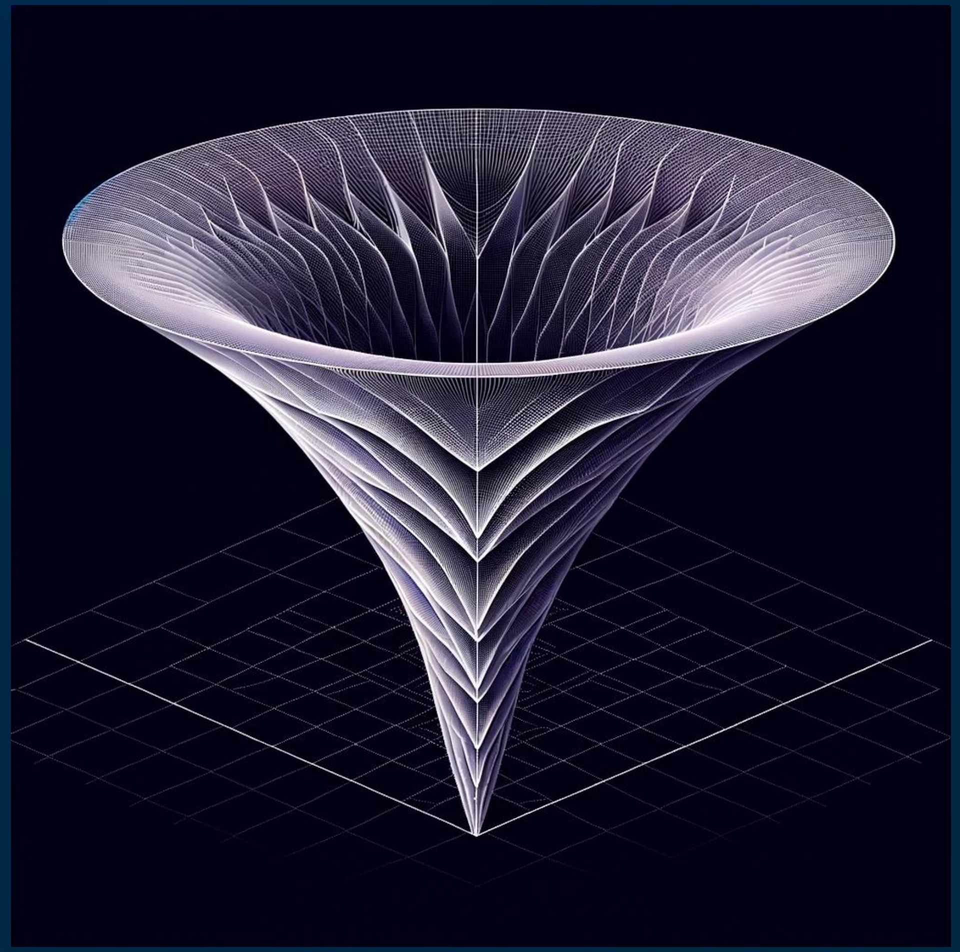
トークプラン

本講演では、宇宙の波動関数に対する新たな枠組みであるローレンツ量子宇宙論(Lorentzian quantum cosmology)を紹介し、量子宇宙論の摂動論的問題について議論する。最初に、このローレンツ量子宇宙論に基づいて、宇宙創生を記述するHartle-Hawkingの無境界仮説とトンネル仮説が厳密に定式化されることを示す。さらに、この量子宇宙論の枠組みで、時空の摂動を考慮すると、無境界仮説とトンネル仮説が宇宙観測と深刻な矛盾を引き起こすことを明らかにし、その摂動的問題がTrans-Planckian物理においても解決できないことを示す。

1. ローレンツ量子宇宙論のReview

2. 量子宇宙論の摂動的問題について議論

3. プランク超物理に基づいた量子宇宙論の摂動的問題の議論



量子宇宙論
（Quantum Cosmology）

量子宇宙論（Quantum Cosmology）

1. Wheeler-DeWitt 方程式（正準量子化）

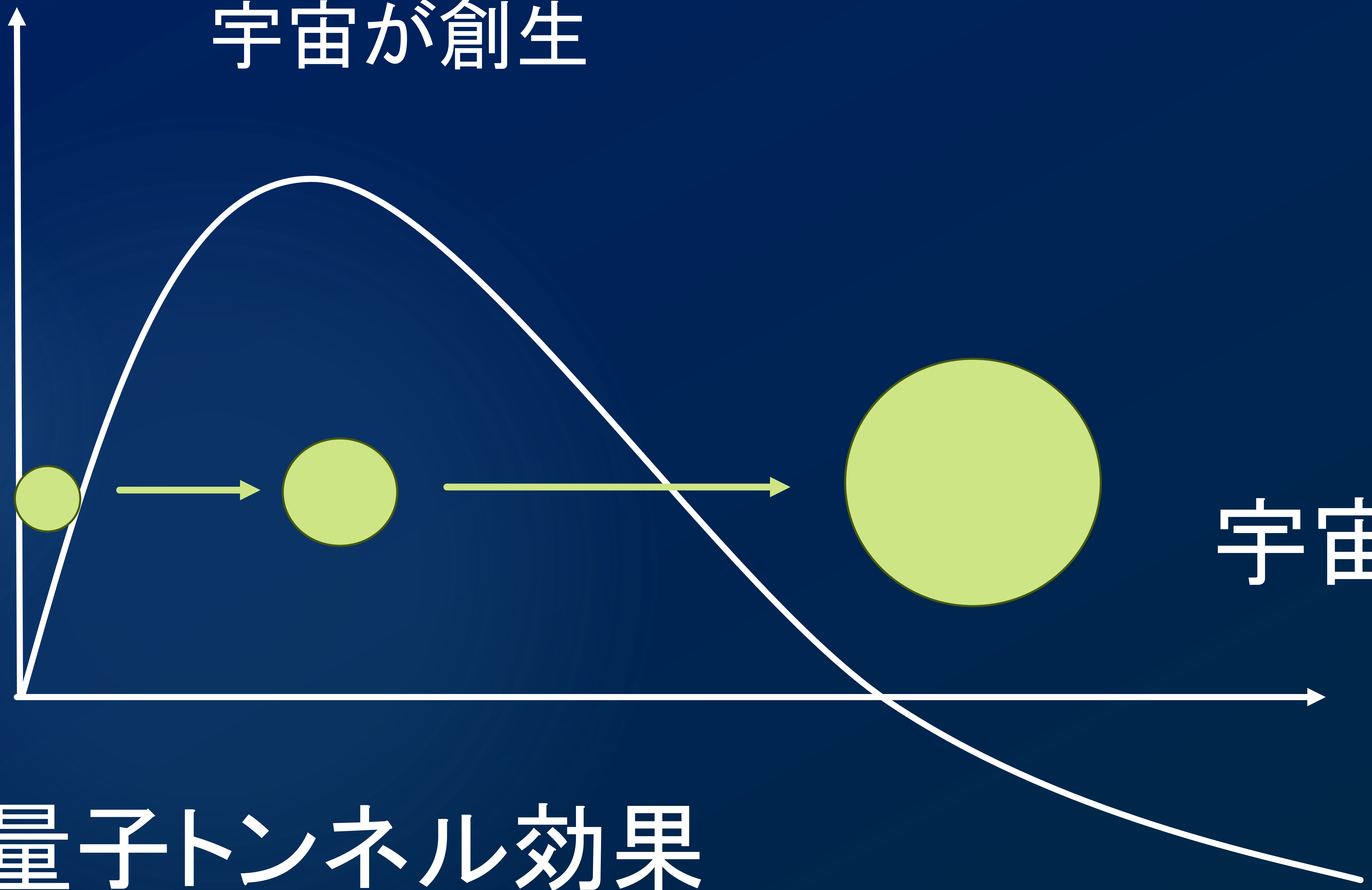
$$\mathcal{H}\Psi = \left[-16\pi G_N \mathcal{G}_{ijkl} \frac{\partial^2}{\partial g_{ij}\partial g_{kl}} + \frac{\sqrt{g}}{16\pi G_N}(-R + 2\Lambda)\right]\Psi = 0$$

2. Path Integral of quantum gravity（経路積分量子化）

$$\Psi(g,\phi) = \int^{(g,\phi)} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi e^{iS[g_{\mu\nu},\phi]/\hbar}$$

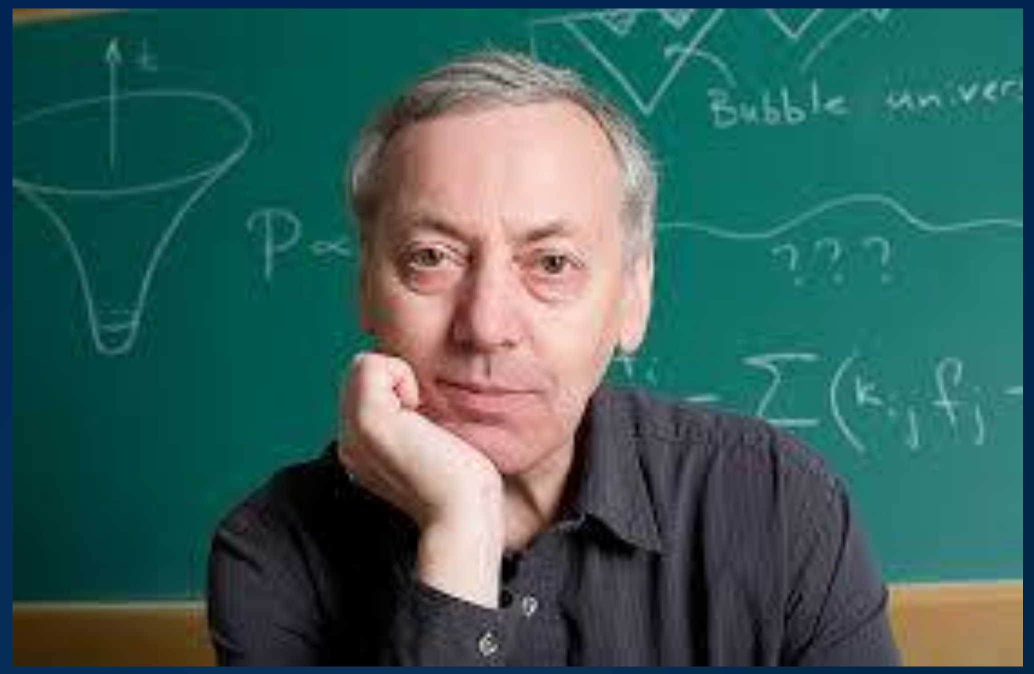
量子宇宙創生

量子トンネル効果により大きさが0の状態から
宇宙が創生



宇宙の大きさ

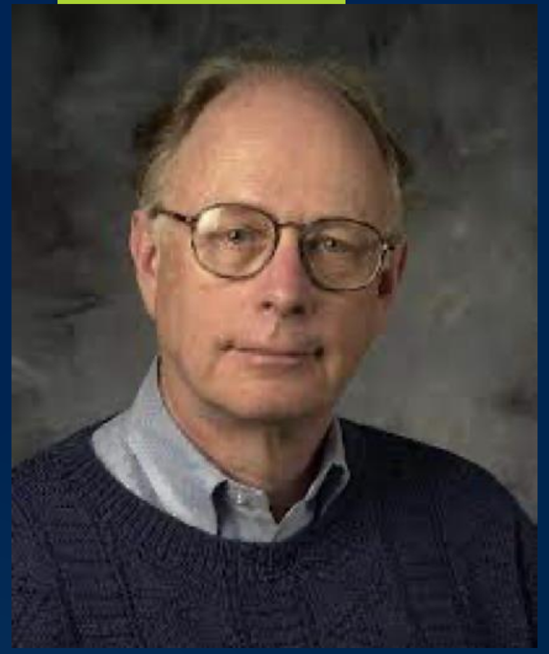
量子トンネル効果



A. Vilenkin

Phys. Lett. B 117 (1982) 25-28.

Hartle-Hawking 無境界仮説



無境界波動関数（無から創成した時空や物質を記述する波動関数）

Phys.Rev.D 28 (1983) 2960-2975

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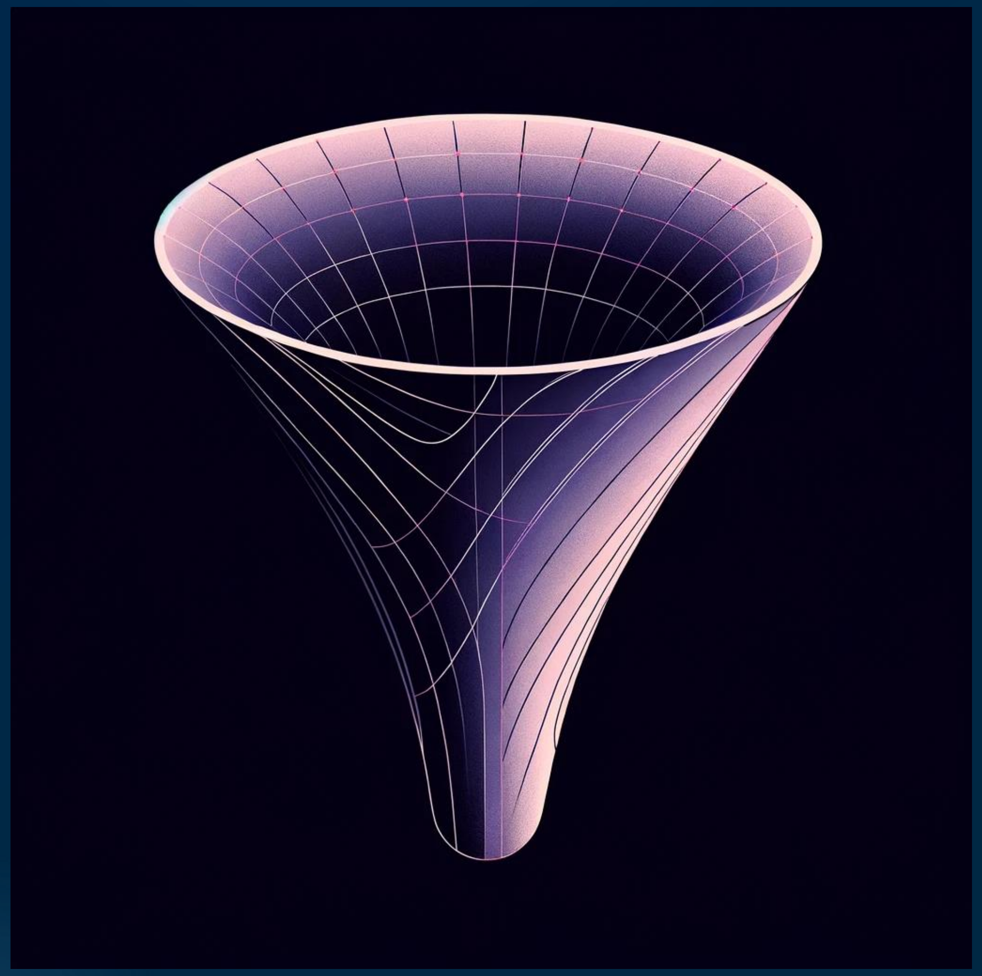
量子重力のユークリッド経路積分
（コンパクトでユークリッド時空を足し上げる）

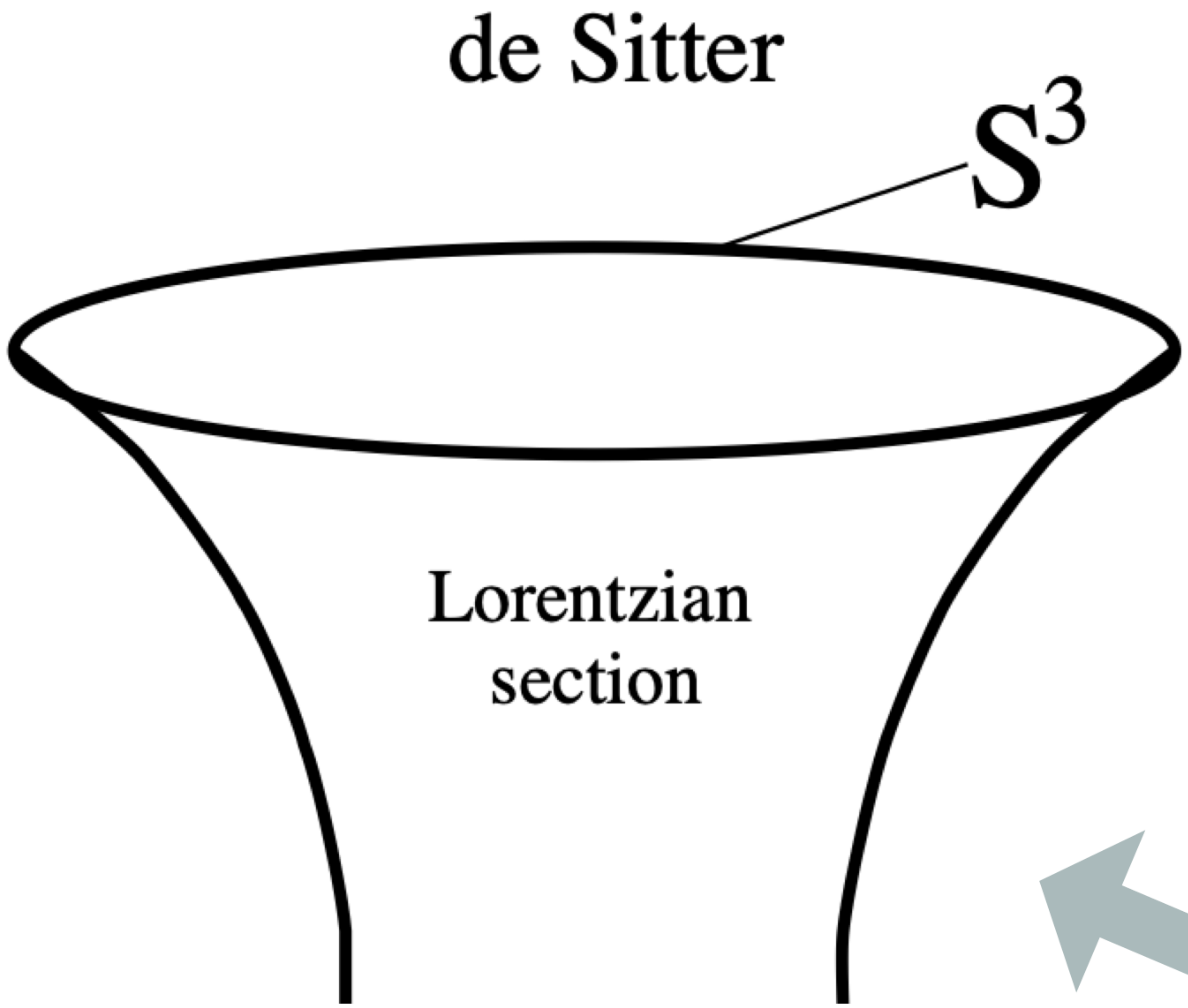
$$\Psi_{\mathrm{HH}} = \int_{\text{no-boundary}}^{(g,\phi)} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \; e^{-S_E[g_{\mu\nu},\phi]/\hbar}$$



Euclidean Action

$$S_{\mathrm{E}} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{g_{\mathrm{E}}}R - i\int \mathrm{d}^3x\sqrt{h_{\mathrm{E}}}K + \int \mathrm{d}^4x\sqrt{g_{\mathrm{E}}}\left(\frac{1}{2}g_{\mathrm{E}}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right)$$

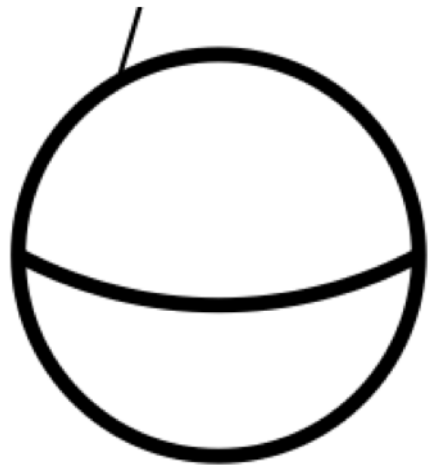
Hartle-Hawking 無境界仮説



R. Bousso, A. Chamblin Phys.Rev.D 59 (1999) 084004

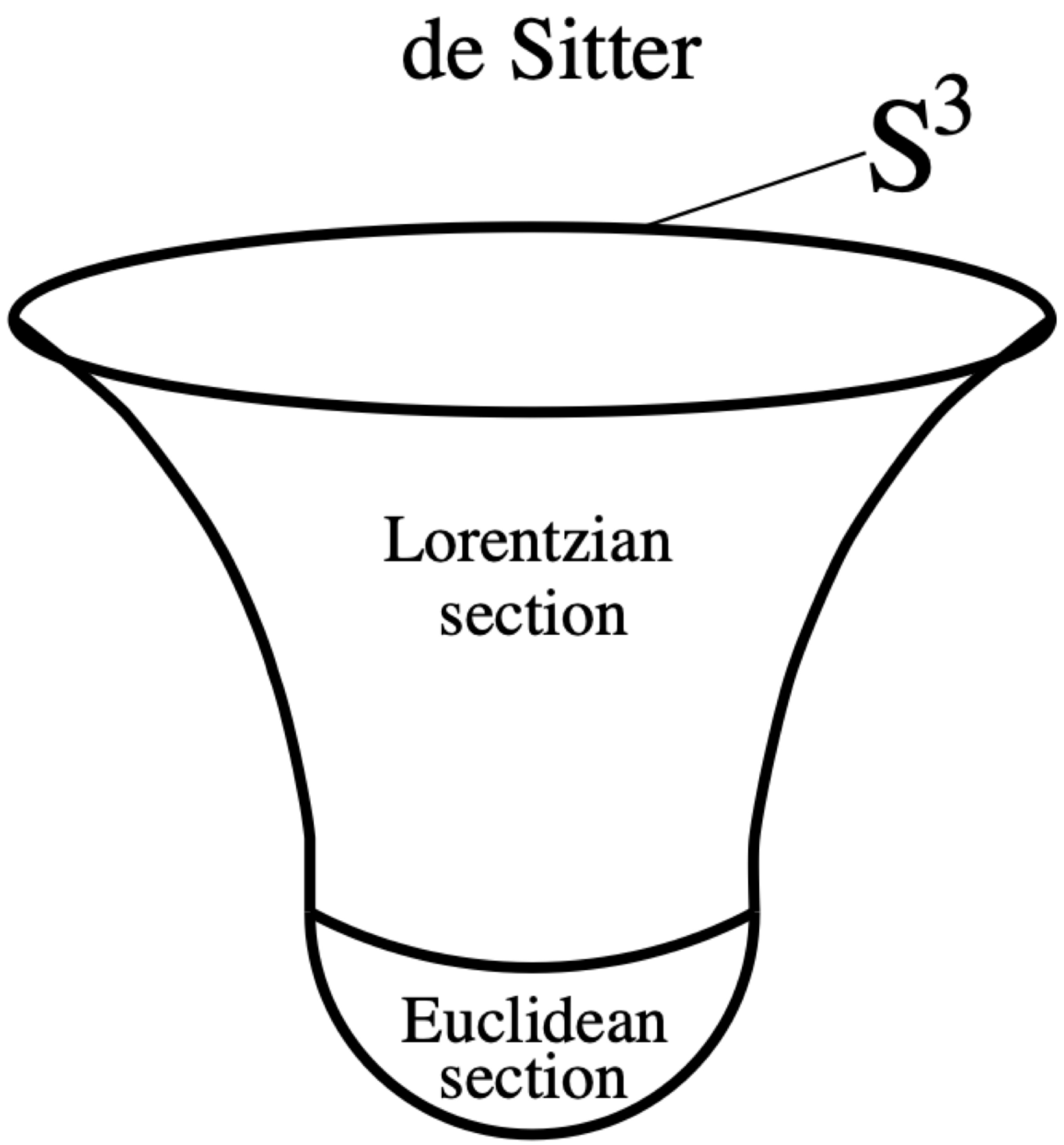
$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} t \ \propto \ e^{\sqrt{\frac{\Lambda}{3}} t}$$

$$\tau = it$$

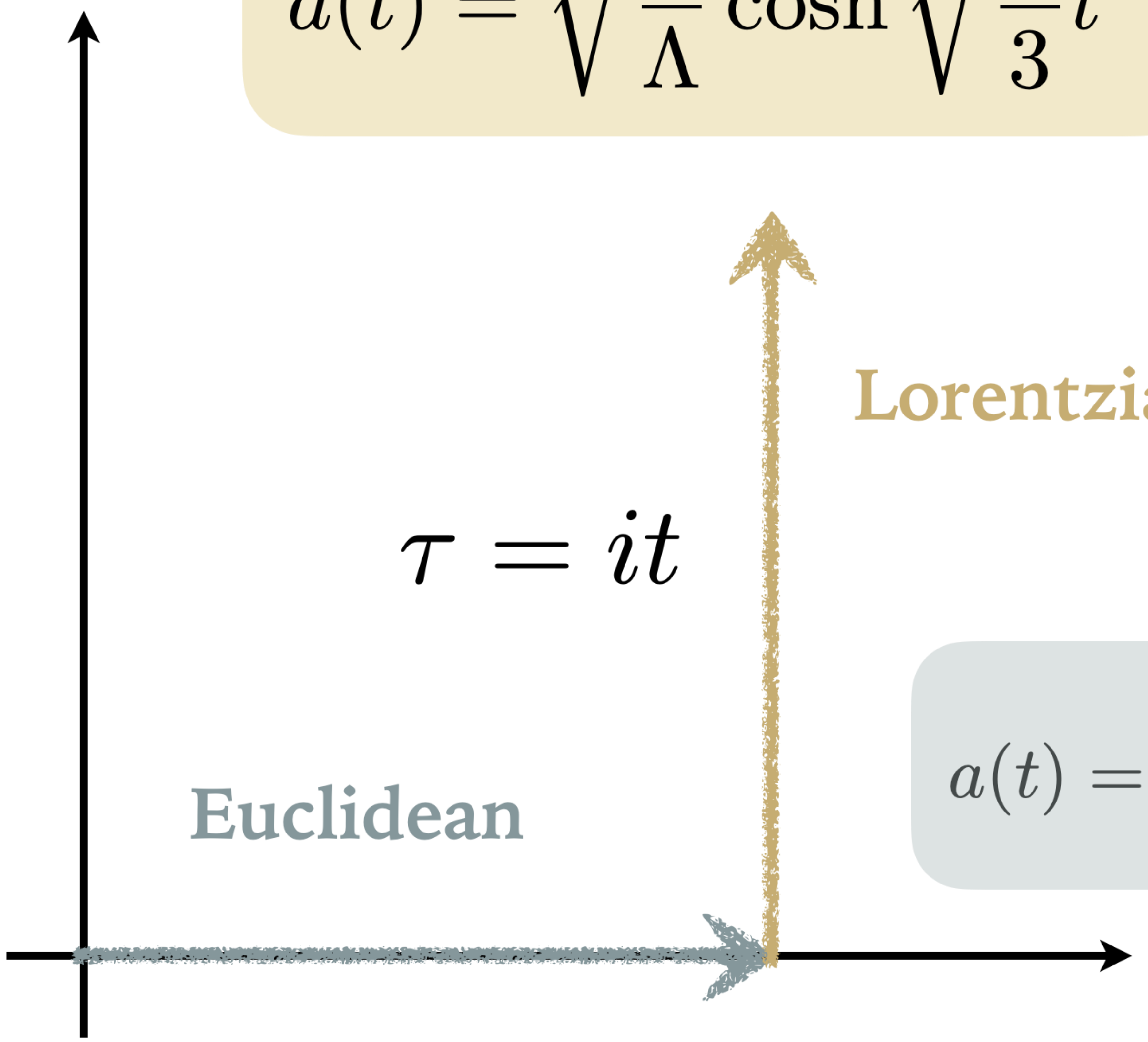


$$a(t) = \sqrt{\frac{3}{\Lambda}} \sin \sqrt{\frac{\Lambda}{3}} \tau$$

Hartle-Hawking 無境界仮説



$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} t$$



$$a(t) = \sqrt{\frac{3}{\Lambda}} \sin \sqrt{\frac{\Lambda}{3}} \tau$$

Conformal factor problem

$$\Psi_{\mathrm{HH}} = \int_{\text{no-boundary}}^{(g,\phi)} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \; e^{-S_E[g_{\mu\nu},\phi]/\hbar}$$

$\Longleftrightarrow$

$$\int \delta a \delta\phi \, e^{-S_E/\hbar}$$
$$\to +\infty$$

Euclidean Solutions

$$\left(\frac{da}{d\tau}\right)^2 - 1 + a^2H^2 = 0, \quad \left(\frac{d^2a}{d\tau^2}\right) = -aH^2.$$

Euclidean On-shell Action

$$S_E[a] = 2\pi^2 \int_0^{\pi/2H} \mathrm{d}\tau \left(-3a\left(\frac{da}{d\tau}\right)^2 - 3a + 3a^3H^2\right) \approx -\frac{12\pi^2}{V(\phi)}$$

No-boundary wave function

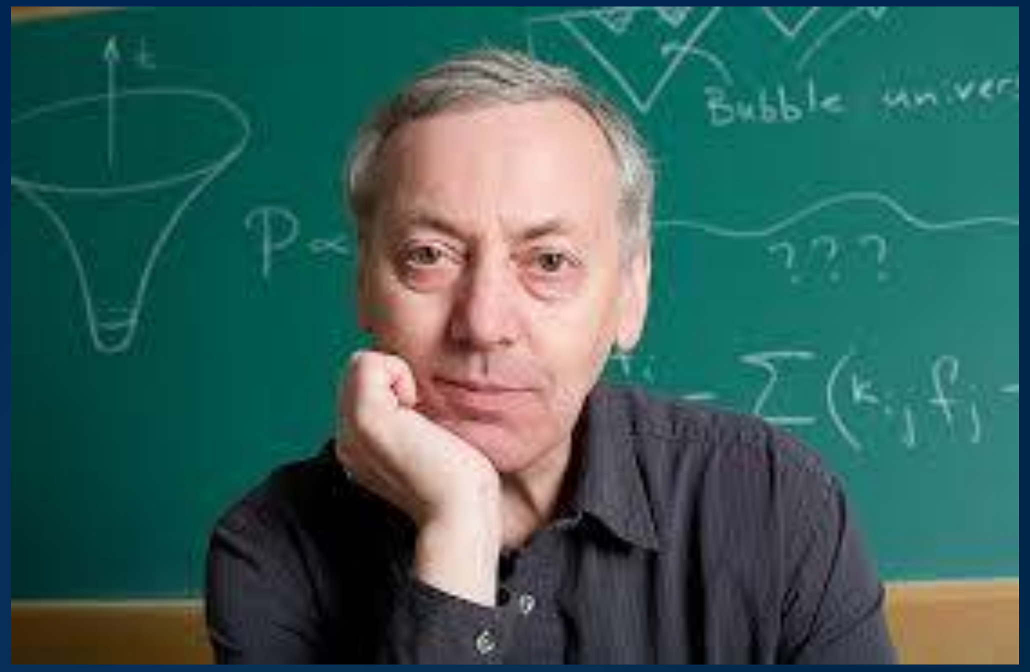
$$\Psi_{\mathrm{HH}}(a,\phi) \sim \exp\left(-S_E[a,\phi]/\hbar\right) \sim \exp\left(+\frac{12\pi^2}{\hbar V(\phi)}\right)$$

Vilenkinのトンネル仮説

量子重力の経路積分

Phys. Rev. D 30 (1984) 509–511

$$\Psi(a,\phi) = \int_{\varnothing}^{(a,\phi)} \mathcal{D}a\,\mathcal{D}\phi\, e^{iS[a,\phi]/\hbar}$$



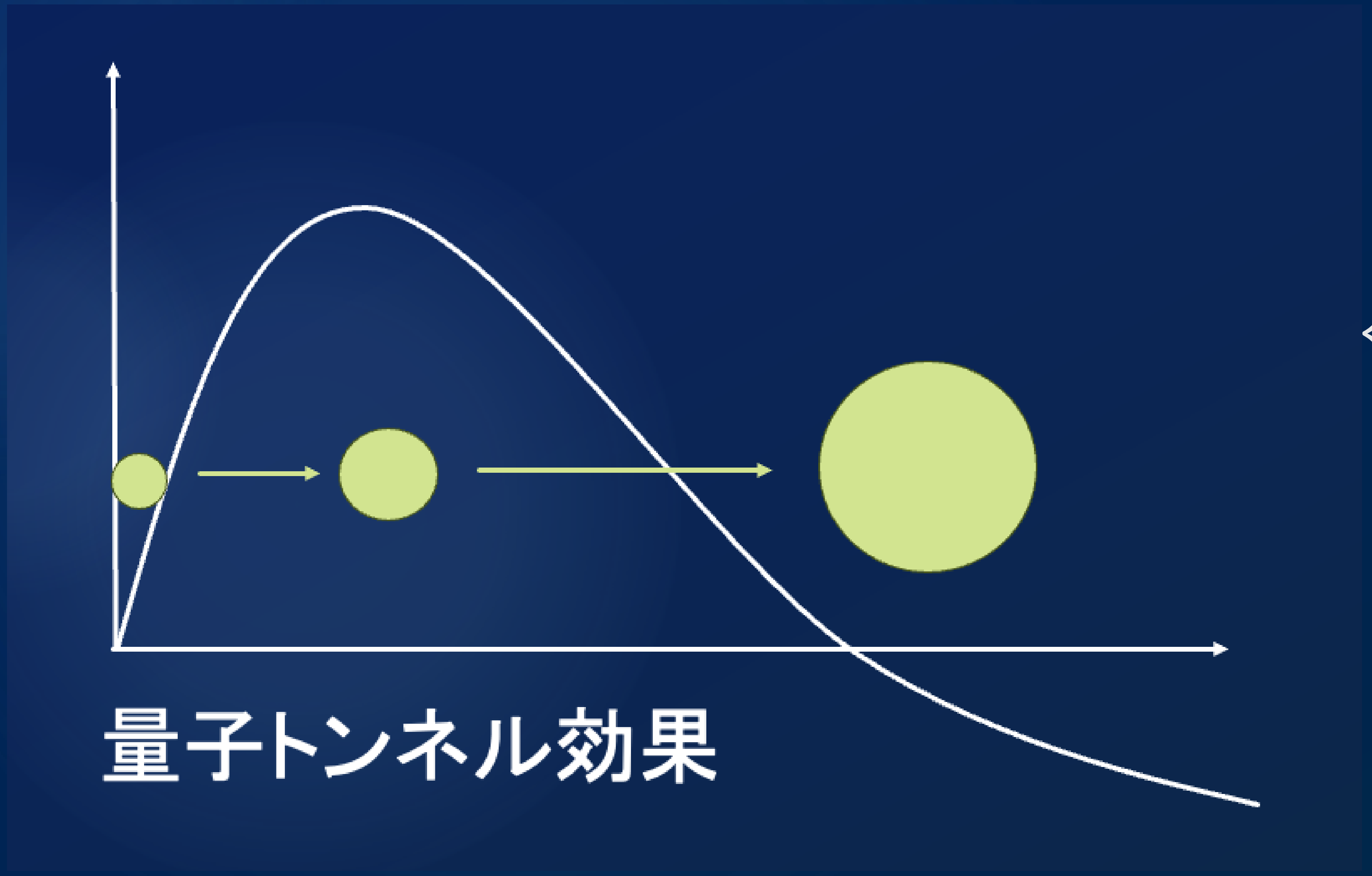
A. Vilenkin

Wheeler-DeWitt 方程式

$$\left\{\frac{1}{12a^2}\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) - \frac{1}{2a^3}\frac{\partial^2}{\partial\phi^2} - U(a,\phi)\right\}\Psi(a,\phi) = 0$$

$$\Psi_T \approx e^{-\frac{12\pi^2}{\hbar V(\phi)}}.$$

$$U(a,\phi) = a^3\left(\frac{3}{a^2} - V(\phi)\right)$$



Hartle-Hawking vs Vilenkin

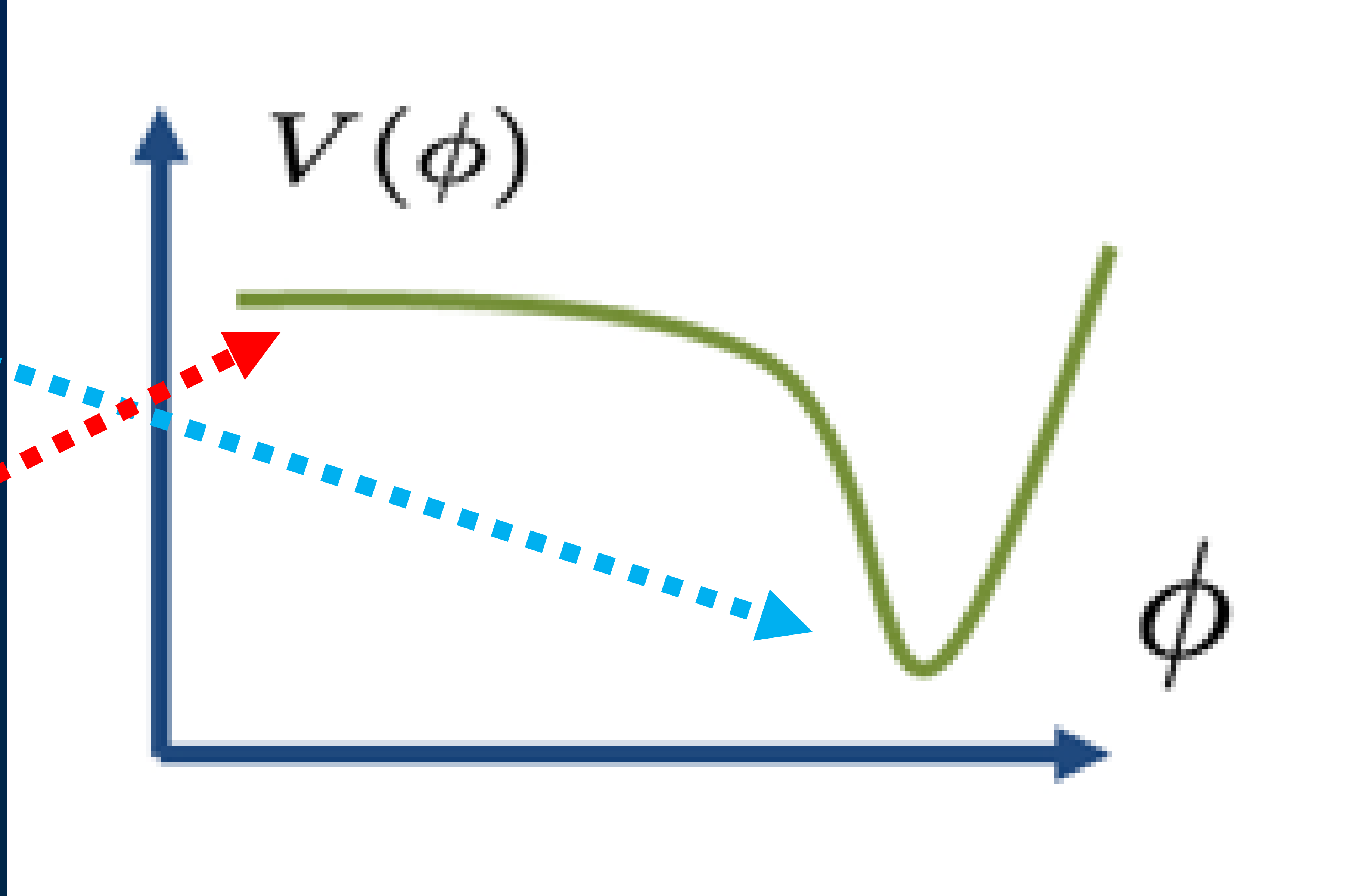
Proceedings of the conference in honor of Stephen Hawking's 60'th birthday
arXiv: gr-qc/0204061

No–boundary wave function

$$\Psi_{HH} \approx e^{+\frac{12\pi^2}{\hbar V(\phi)}}.$$

Tunneling wave function

$$\Psi_{T} \approx e^{-\frac{12\pi^2}{\hbar V(\phi)}}.$$



Lorentzian Quantum Cosmology

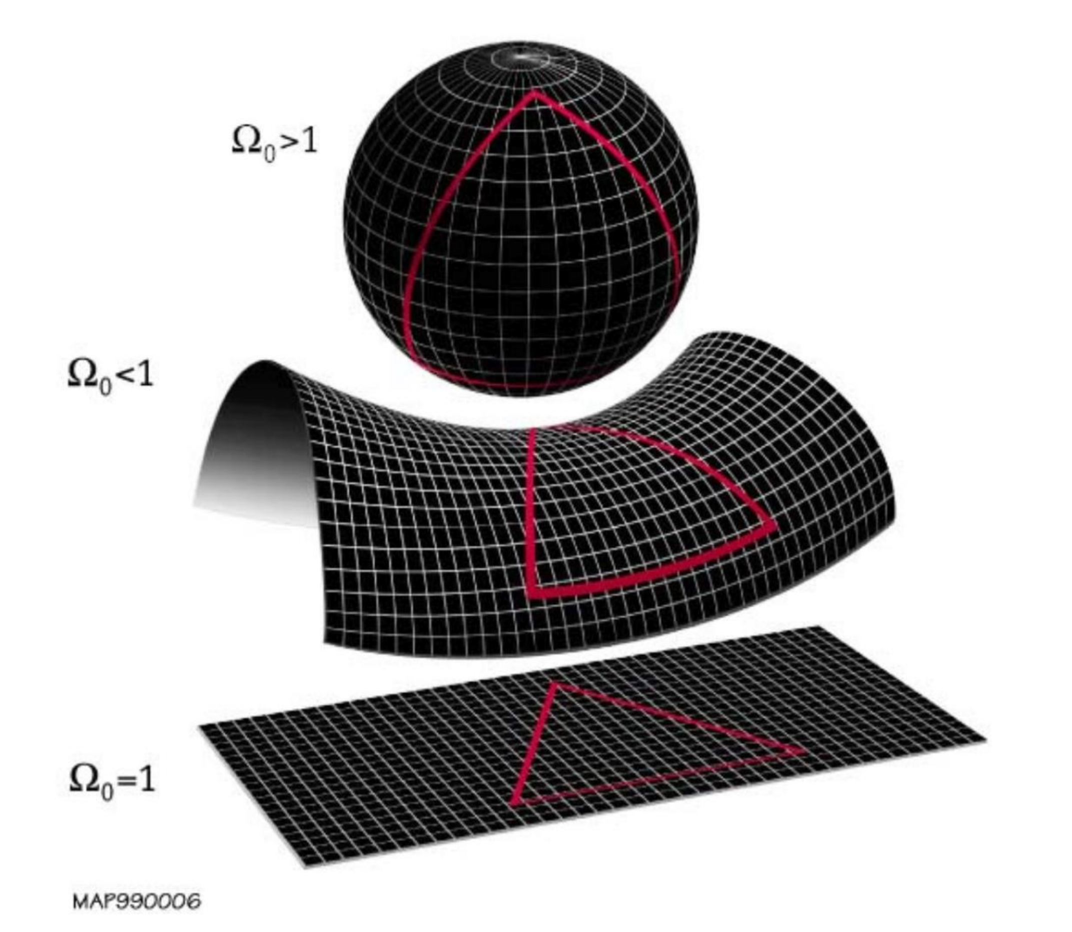
Our setup

$$ds^2 = -\frac{N^2(t)}{q(t)}dt^2 + q(t)\left[\Omega_{ij}(\mathbf{x}) + h_{ij}(t\,,\mathbf{x})\right] dx^i dx^j$$

Lapse function

Scale factor

Tensor fields



Gravitational action is expanded up to the second order in the perturbation

$$S_{\rm GR}^{(0)} = V_3 \int_{t=t_i}^{t=t_f} dt \left(-\frac{3}{4N}\dot{q}^2 + N(3K - \Lambda q)\right) + S_B\,,$$

$$S_{\rm GR}^{(2)} = V_3 \int_{t=t_i}^{t=t_f} N dt \sum_{snlm}\left[\frac{q^2}{8N^2}\left(\dot{h}^s_{nlm}\right)^2 - \frac{K}{8}\left((n^2-3)+2\right)\left(h^s_{nlm}\right)^2\right],$$

$$M_{\rm Pl} = 1/\sqrt{8\pi G} = 1$$

$$V_3 = 2\pi^2$$

$$h_{ij}(t\,,\mathbf{x}) = \sum_{snlm} h^s_{nlm}(t) Q^{snlm}_{ij}$$

$$n \geq 3,\ l \in [0, n-1],\ m \in [-l, l]$$

$$K = +1$$

Batalin-Fradkin-Vilkovisky (BFV) formalism

Path Integral of Quantum Gravity

$$G[q,h] \equiv \int \mathcal{D}q\mathcal{D}h\mathcal{D}p_q\mathcal{D}p_h\mathcal{D}\Pi\,\mathcal{D}N\,\mathcal{D}\rho\,\mathcal{D}\bar{c}\,\mathcal{D}\bar{\rho}\,\mathcal{D}c\,\exp(iS_{\mathrm{BRS}}/\hbar)$$

Becchi-Rouet-Stora (BRS) invariant action

$$S_{\mathrm{BRS}} \equiv \int_{t_i}^{t_f} \mathrm{d}t\left(p_q\dot{q} + p_h\dot{h} - N\mathcal{H} + \Pi\dot{N} + \bar{\rho}\dot{c} + \bar{c}\dot{\rho} - \bar{\rho}\rho\right)$$

Lagrange multiplier and ghost fields

BRS symmetry transformation

$$\delta a = \lambda c\frac{\partial\mathcal{H}}{\partial p_a}\,,\quad \delta p_a = -\lambda c\frac{\partial\mathcal{H}}{\partial a}\,,\quad \delta h = \lambda c\frac{\partial\mathcal{H}}{\partial p_h}\,,\quad \delta p_h = -\lambda c\frac{\partial\mathcal{H}}{\partial h}\,,$$

$$\delta N = \lambda\rho\,,\quad \delta\bar{c} = -\lambda\Pi\,,\quad \delta\bar{\rho} = -\lambda\mathcal{H}\,,\quad \delta\Pi = \delta c = \delta\rho = 0$$

Lorentzian path integral

$$\begin{aligned} G[q,h] &= \int \mathrm{d}N(t_f - t_i)\int \mathcal{D}q\mathcal{D}h\mathcal{D}p_q\mathcal{D}p_h\exp\left(i\int_{t_i}^{t_f}\mathrm{d}t\left(p_q\dot{q} + p_h\dot{h} - N\mathcal{H}\right)/\hbar\right) \\ &= \int \mathrm{d}N(t_f - t_i)\int \mathcal{D}q\mathcal{D}h\exp\left(iS_{\mathrm{GR}}[q,h,N]/\hbar\right) \end{aligned}$$

Lorentzian Quantum Cosmology

Lorentzian path integral for no-boundary and tunneling proposal

$$G^{(0)}[q_f] = \int \mathrm{d}N \int_{q(t=t_i)=0}^{q(t=t_f)=q_f} \mathcal{D}q \exp\left(iS_{\mathrm{GR}}^{(0)}[N,q]/\hbar\right)$$

Action $S_{\mathrm{GR}}^{(0)}[N,q] = V_3 \int_{t_i=0}^{t_f=1} \mathrm{d}t \left(-\frac{3}{4N}\dot{q}^2 + N(3K - \Lambda q)\right)$

Scalar factor $q(t) = a(t)^2 \quad H^2 = \frac{\Lambda}{3}$

Boundary conditions

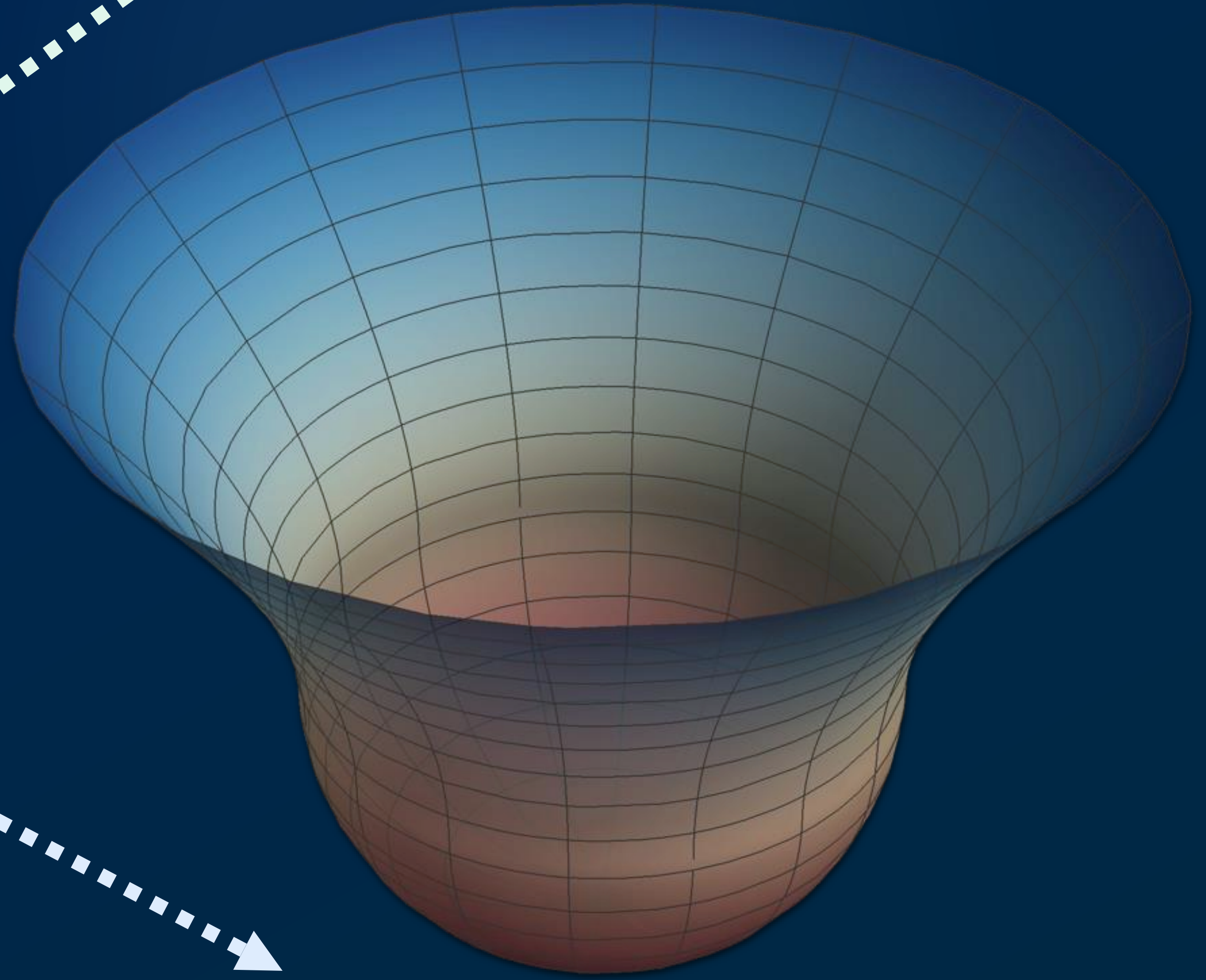
$$q(t_i = 0) = 0, \quad q(t_f = 1) = q_f$$

EOM $\frac{\delta S_{\mathrm{GR}}^{(0)}[q]}{\delta q} = 0 \implies \frac{1}{2N^2}\ddot{q} = H^2$

$$q_s = H^2 N^2 t^2 + \left(-H^2 N^2 + q_f\right) t$$

On-shell action $S_{\text{on-shell}}^{(0)}[N] = V_3 \left[\frac{N^3 H^4}{4} + N\left(-\frac{3H^2}{2}(q_f) + 3K\right) + \frac{1}{N}\left(-\frac{3}{4}(q_f)^2\right)\right]$

$q(t = t_f) = q_f$

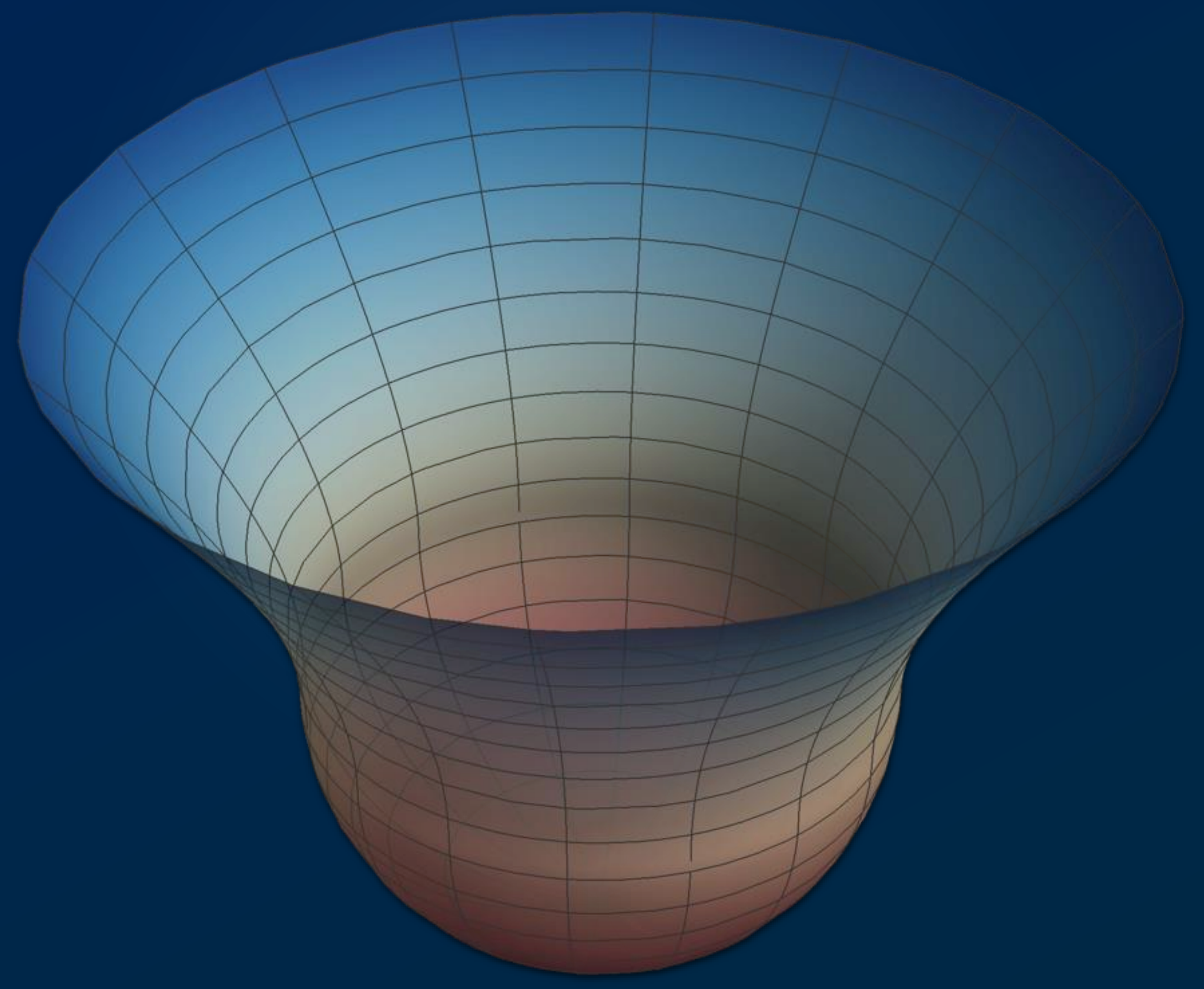


$q(t = t_i) = 0$

Lorentzian Quantum Cosmology

Lorentzian path integral for no-boundary and tunneling proposal

$$G^{(0)}[q_f] = \int \mathrm{d}N \int_{q(t=t_i)=0}^{q(t=t_f)=q_f} \mathcal{D}q\, e^{iS^{(0)}_{\mathrm{GR}}[N,q]/\hbar}$$
$$= \int \mathrm{d}N\, e^{iS^{(0)}_{\text{on-shell}}[N]/\hbar} \int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Q(t)\, e^{iS_Q/\hbar},$$



Quantum fluctuations $q(t) = q_s(t) + Q(t) \quad q_s = \frac{\Lambda}{3}N^2t^2 + \left(-\frac{\Lambda}{3}N^2 + q_f\right)t$

Path integral over Q(t) can be evaluated

$$\int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Q(t) e^{iS_Q/\hbar} = \int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Q(t) \exp\left(-\frac{3iV_3}{4N\hbar}\int_0^1 \mathrm{d}t\,\dot{Q}^2\right) = \sqrt{\frac{3iV_3}{4\pi N\hbar}}$$

On-shell action

$$G^{(0)}[q_f] = \sqrt{\frac{3iV_3}{4\pi\hbar}} \int_0^\infty \frac{\mathrm{d}N}{N^{1/2}} e^{iS^{(0)}{}_{\text{on-shell}}[N]/\hbar}$$

$$S^{(0)}_{\text{on-shell}}[N] = V_3\left[\frac{N^3H^4}{4} + N\left(-\frac{3H^2}{2}(q_f) + 3K\right) + \frac{1}{N}\left(-\frac{3}{4}(q_f)^2\right)\right]$$

Picard-Lefschetz theory

$$G^{(0)}[q_f] = \sqrt{\frac{3iV_3}{4\pi\hbar}} \int_0^\infty \frac{\mathrm{d}N}{N^{1/2}} e^{iS^{(0)}{}_{\text{on-shell}}[N]/\hbar}$$

$$S^{(0)}_{\text{on-shell}}[N] = V_3 \left[\frac{N^3H^4}{4} + N\left(-\frac{3H^2}{2}(q_f) + 3K\right) + \frac{1}{N}\left(-\frac{3}{4}(q_f)^2\right) \right]$$

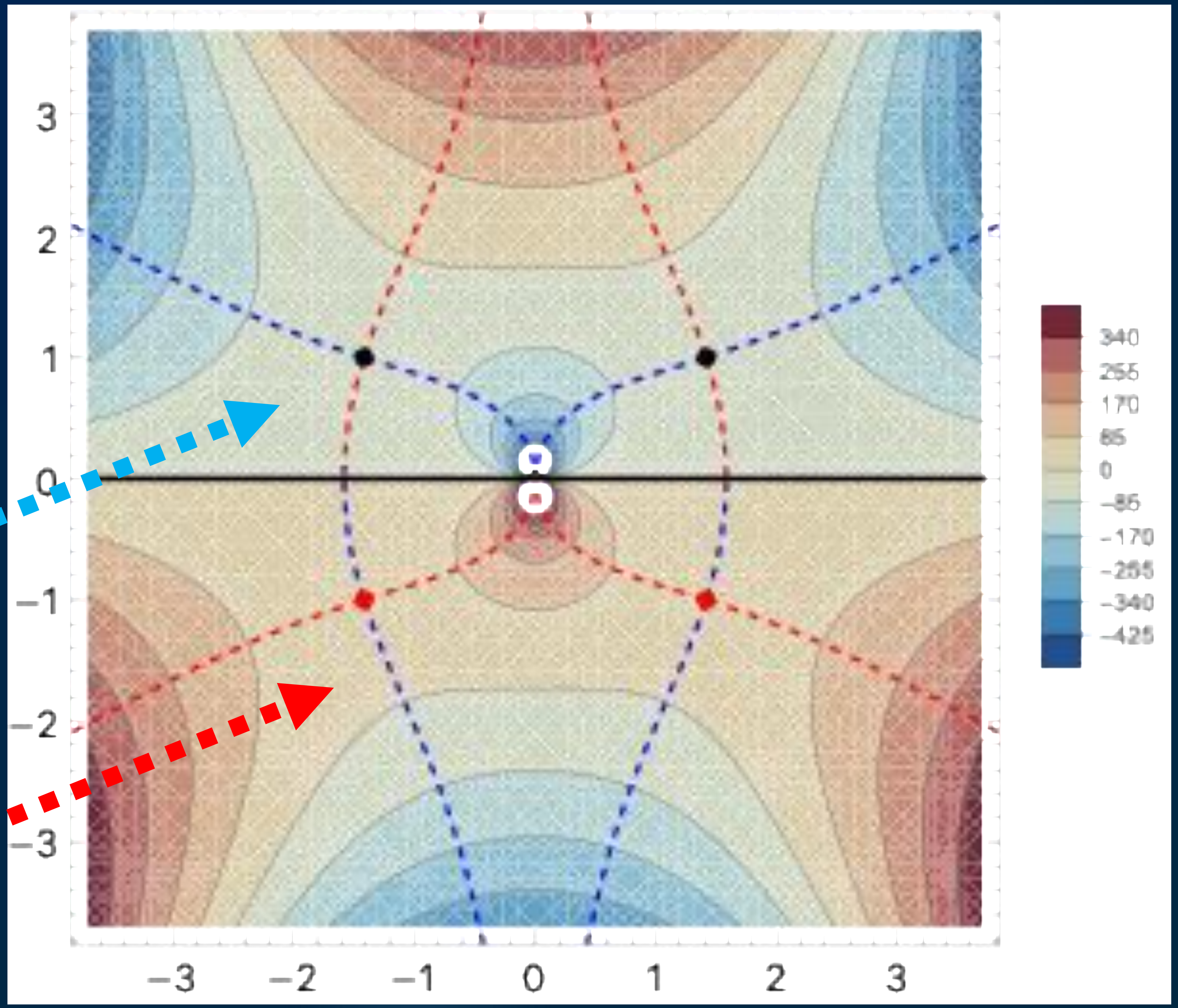
Picard-Lefschetz theory

$$\int_{\mathcal{R}} \mathrm{d}x\, e^{if(x)} \implies \int_{\mathcal{C}} \mathrm{d}x\, e^{if(x)} = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}z\, e^{if(z)}$$

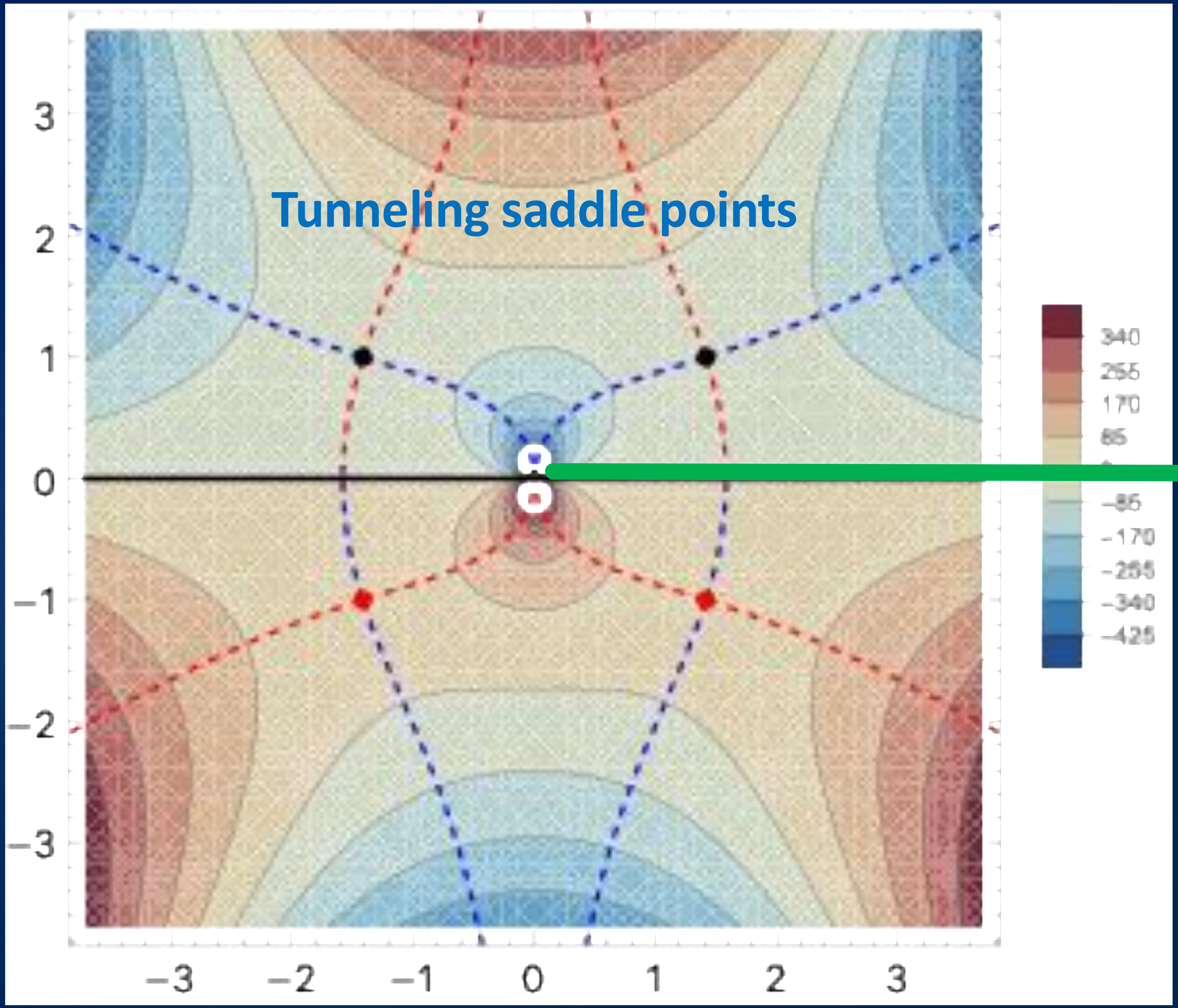
Steepest descent (Lefschetz thimble)

Tunneling saddle points $N_{\mathrm{T}} = \frac{1}{H^2}\left[i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$

No-boundary saddle points $N_{\mathrm{HH}} = \frac{1}{H^2}\left[-i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$



Integration Contour (0<N<+∞)



J. Feldbrugge, J.-L. Lehners and N. Turok,
Phys. Rev. D 95 (2017) 103508

1 (tunneling) saddle point

$$N_s = \frac{1}{H^2}\left[i + \left(q_f H^2 - 1\right)^{1/2}\right]$$

Tunneling wave function

$$\Psi_T \sim e^{-\frac{12\pi^2}{\hbar\Lambda}}$$

Integration Contour of Lapse function

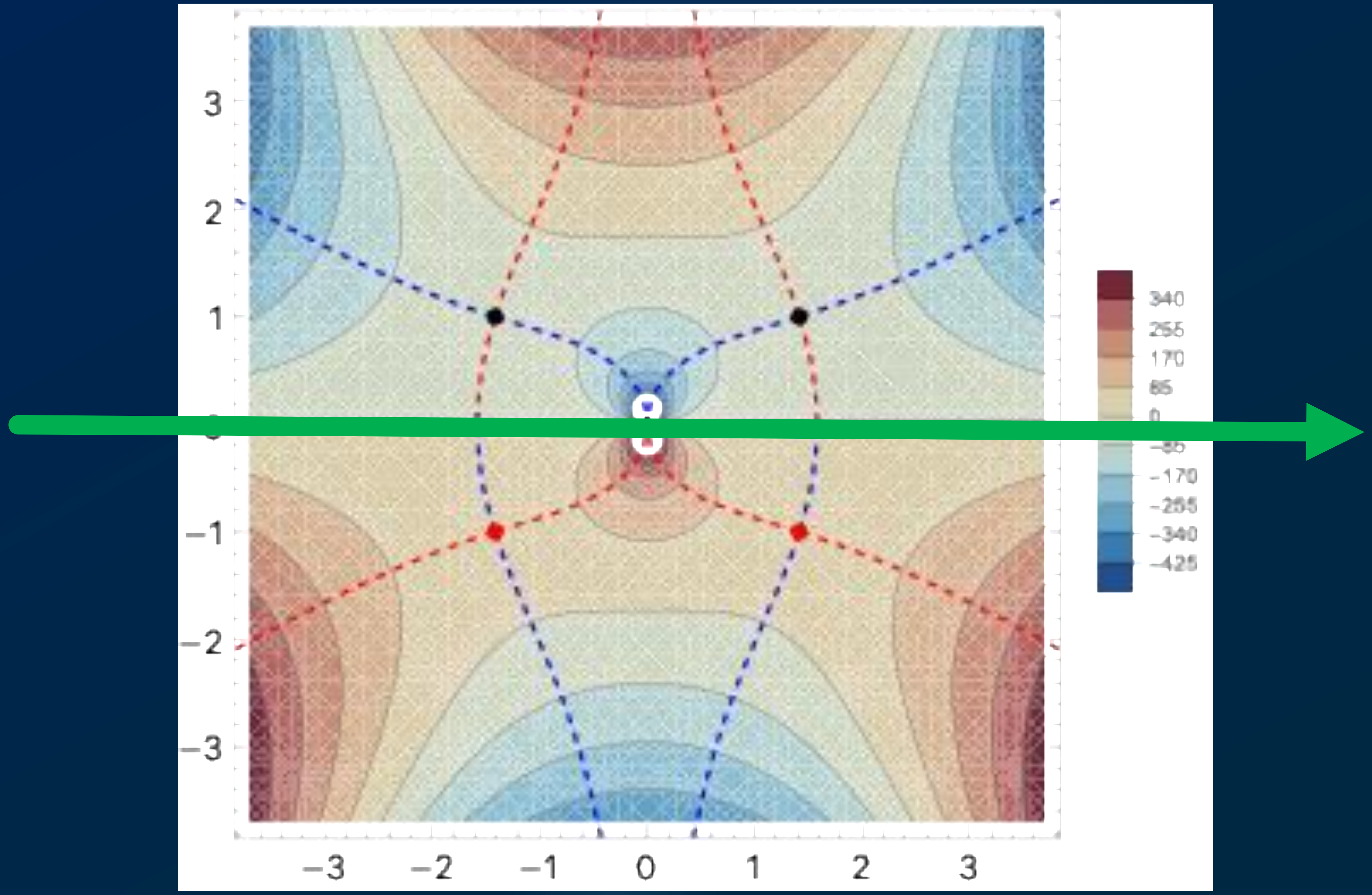
J. Diaz Dorronsoro, J. J. Halliwell, J. B. Hartle, T. Hertog and O. Janssen, Phys. Rev. D 96 (2017) 043505

$$N > 0 \qquad d\tau = N dt$$

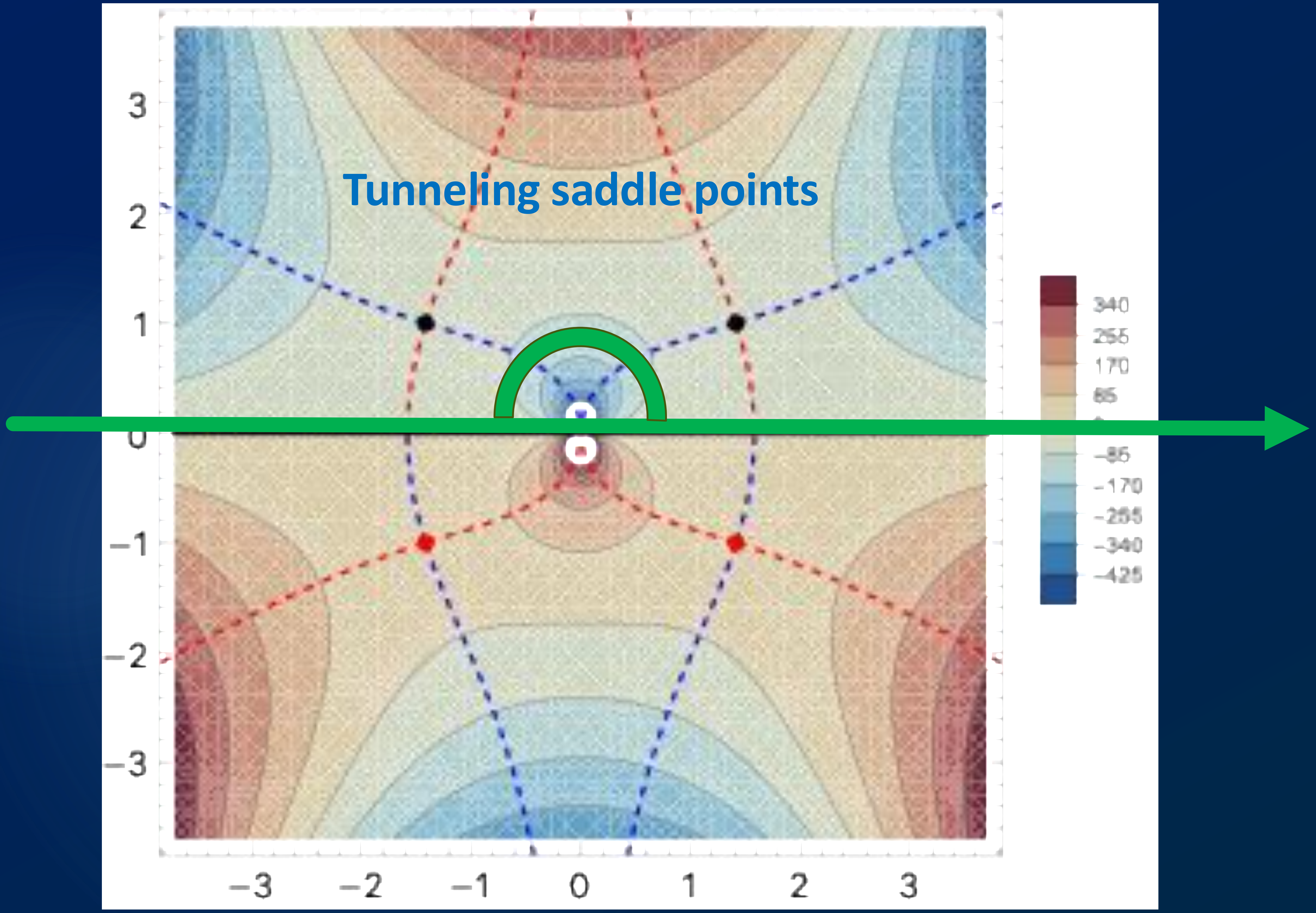
$$G[q,h] = \int_0^\infty \mathrm{d}N \int \mathcal{D}q\mathcal{D}h \exp\left(iS_{\mathrm{GR}}[N,q,h]/\hbar\right)$$

$$-\infty < N < +\infty$$

$$G[q,h] = \int_{-\infty}^\infty \mathrm{d}N \int \mathcal{D}q\mathcal{D}h \exp\left(iS_{\mathrm{GR}}[N,q,h]/\hbar\right)$$



Integration Contour 1 (-∞<N<+∞)



Phys.Rev.D 97 (2018) 2, 023509

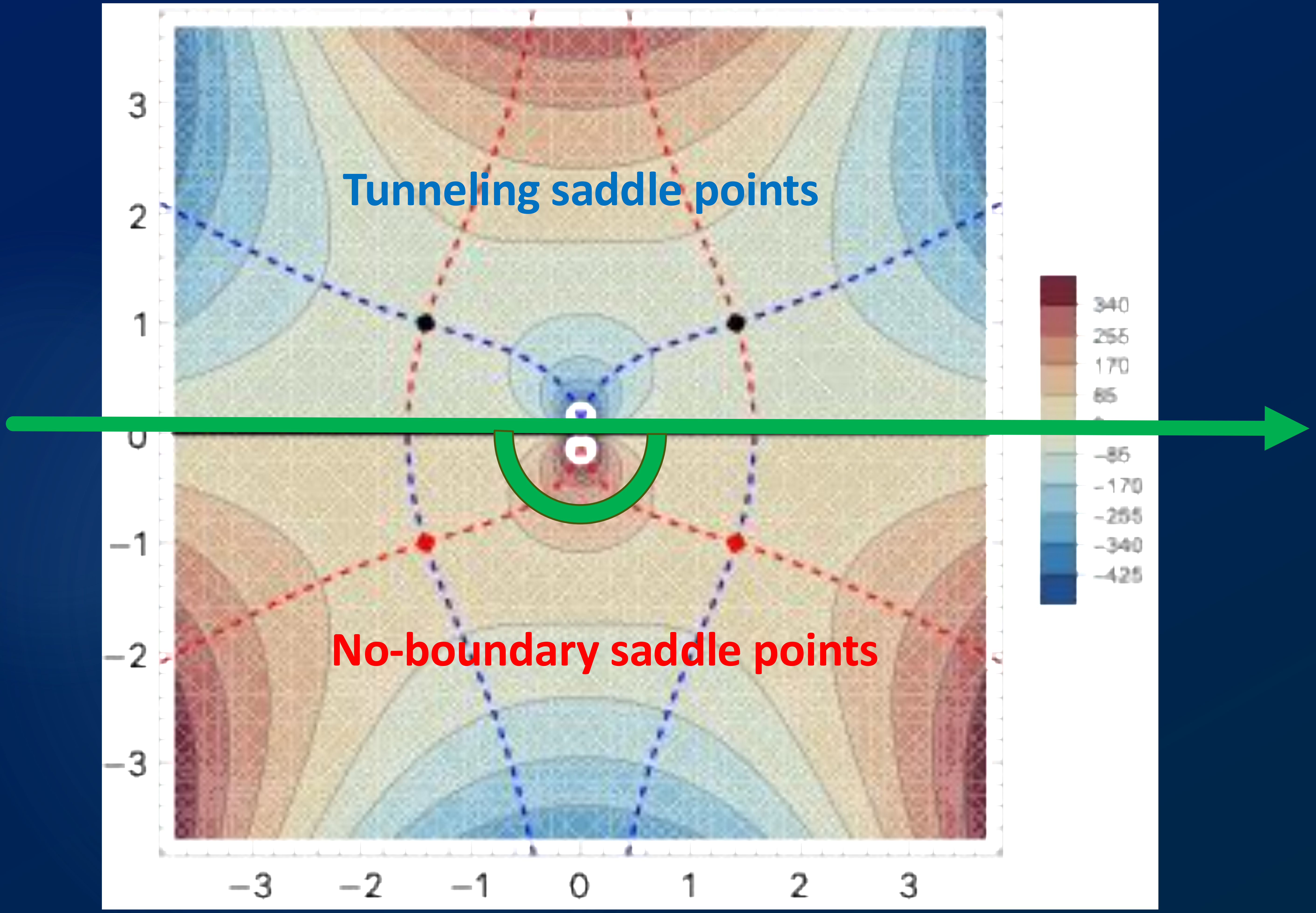
2 (tunneling) saddle points

$$N_s = \frac{1}{H^2}\left[i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$$

Tunneling wave function

$$\Psi_T \sim e^{-\frac{12\pi^2}{\hbar\Lambda}}$$

Integration Contour 2 (-∞<N<+∞)



Phys. Rev. D 96 (2017) 043505,

Phys.Rev.D 97 (2018) 2, 023509

4 (all) saddle points

$$N_s = \frac{1}{H^2}\left[\pm i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$$

No-boundary wave function

$$\Psi_{HH} \sim e^{+\frac{12\pi^2}{\hbar\Lambda}}$$

Picard-Lefschetz and Resurgence Analysis

$$G^{(0)}[q_f] = \sqrt{\frac{3iV_3}{4\pi\hbar}} \int_0^\infty \frac{\mathrm{d}N}{N^{1/2}} e^{iS^{(0)}{}_{\text{on-shell}}[N]/\hbar}$$

[M. Honda, H. Matsui, K. Okabayashi, T. Terada, 2402.09981]

$$S^{(0)}_{\text{on-shell}}[N] = V_3 \left[\frac{N^3H^4}{4} + N\left(-\frac{3H^2}{2}(q_f) + 3K\right) + \frac{1}{N}\left(-\frac{3}{4}(q_f)^2\right)\right]$$

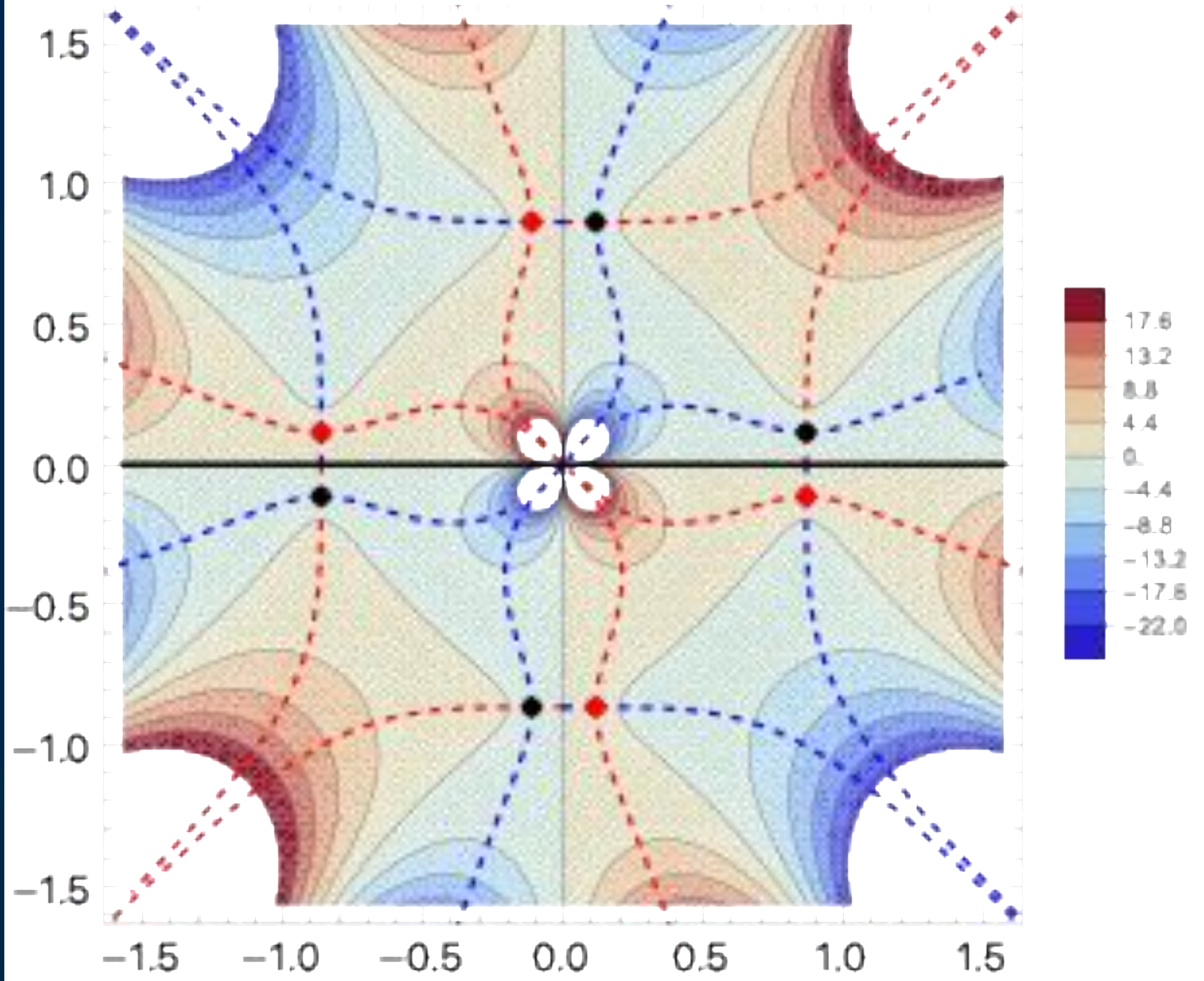
$\Updownarrow$ $$N = x^2$$

$$12\alpha\gamma + \beta^2 < 0$$

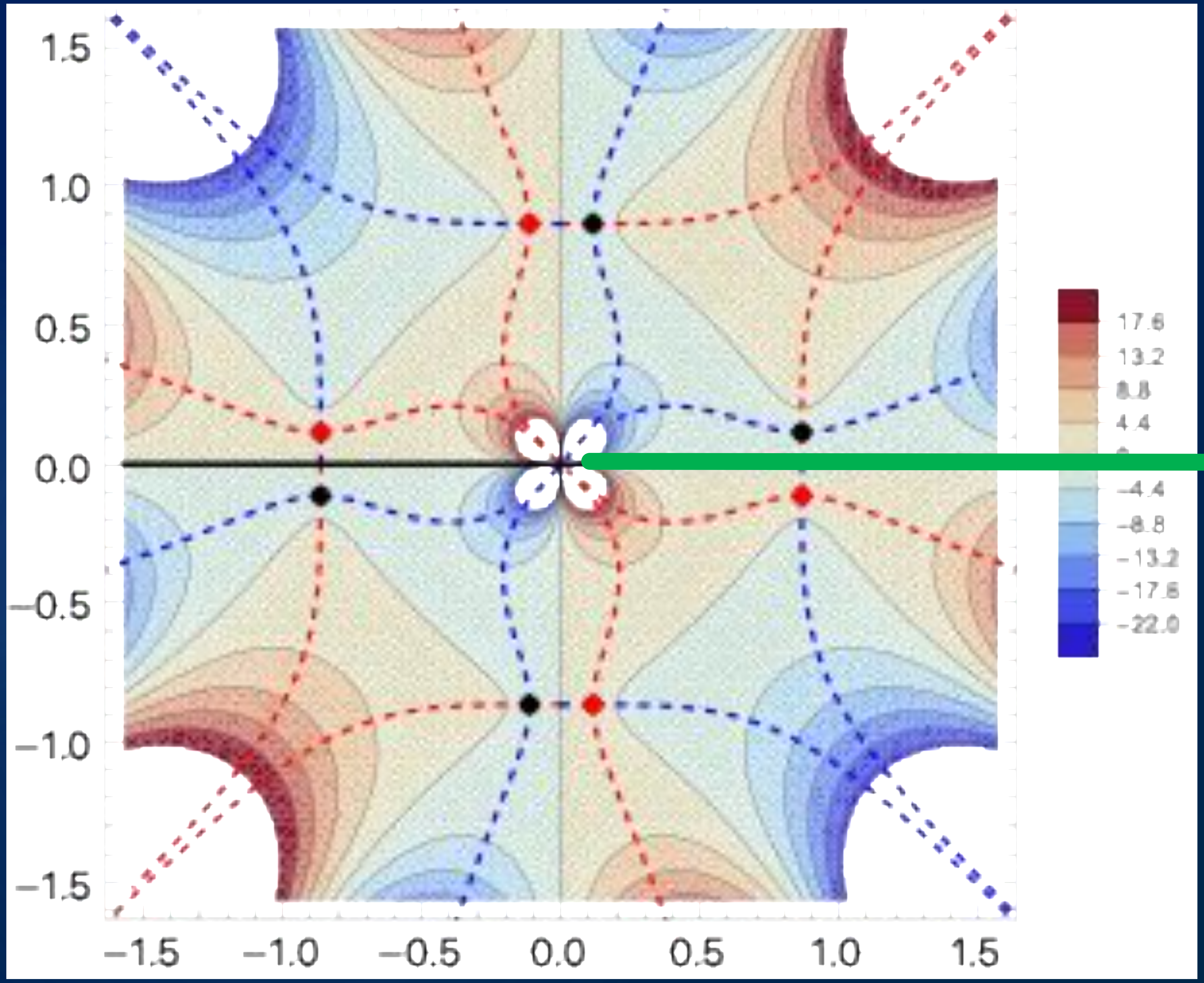
$$G(\hbar) = \sqrt{\frac{3iV_3}{4\pi\hbar}} \int_{-\infty}^\infty \mathrm{d}x \exp[F(x)]$$

$$F(x) := \frac{i}{\hbar} S_{\text{on-shell}}(x) \qquad S_{\text{on-shell}}(x) = \alpha x^6 + \beta x^2 + \frac{\gamma}{x^2}$$

$$\alpha = V_3 \frac{\Lambda^2}{36}, \quad \beta = V_3\left(-\frac{\Lambda(q_i + q_f)}{2} + 3K\right), \quad \gamma = V_3\left(-\frac{3}{4}(q_f - q_i)^2\right)$$



Integration Contour of x



$$12\alpha\gamma + \beta^2 < 0$$

No-boundary and tunneling saddles are on Stokes lines (thimble decomposition is ambiguous) . We can not determine whether tunneling saddle points or no-boundary saddle points contribute

$$F(x) := \frac{i}{\hbar} S_{\text{on-shell}}(x)$$

We deform the hbar to be complex

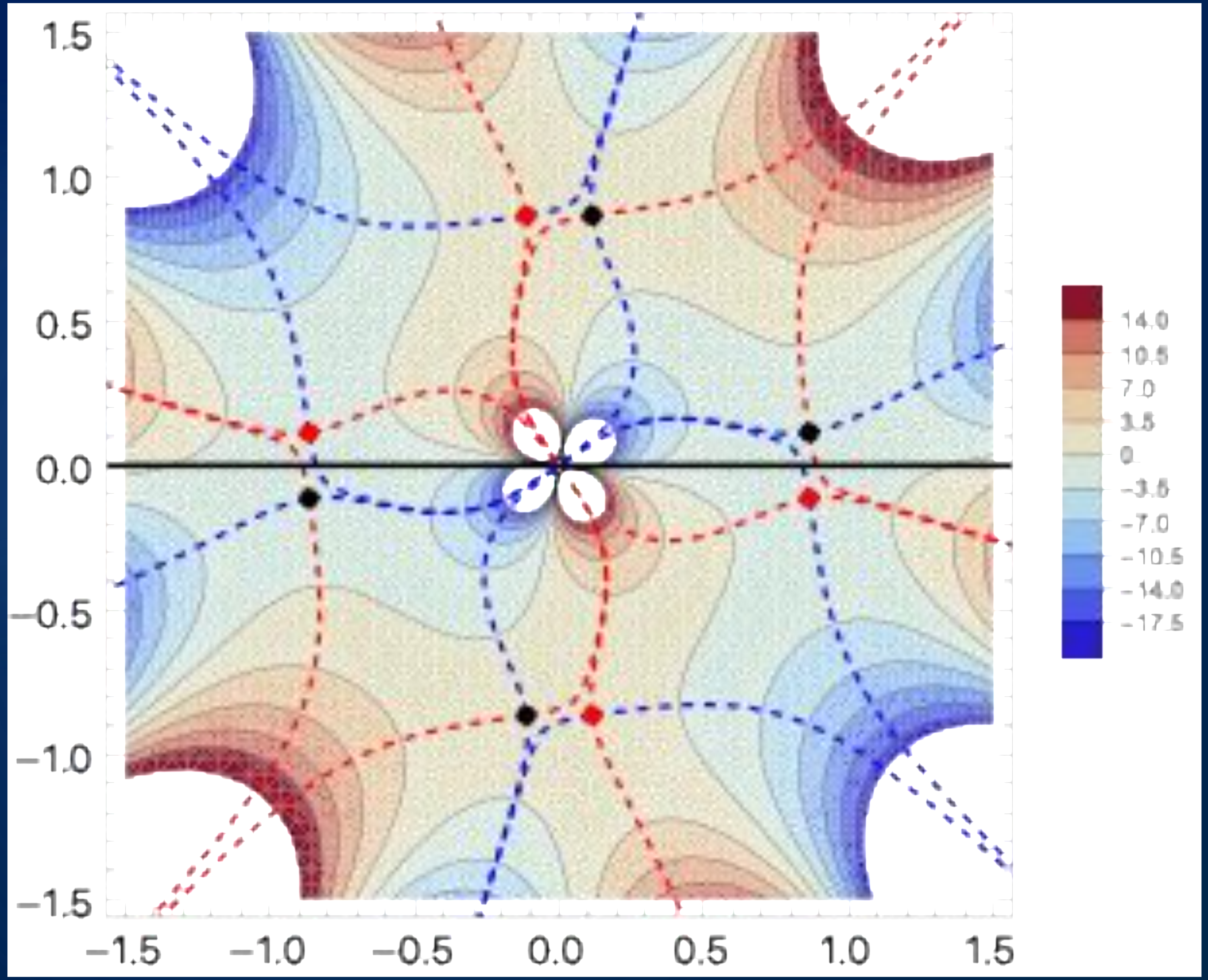
$$\hbar = e^{i\theta}|\hbar| \qquad \theta \to 0_{\pm}$$

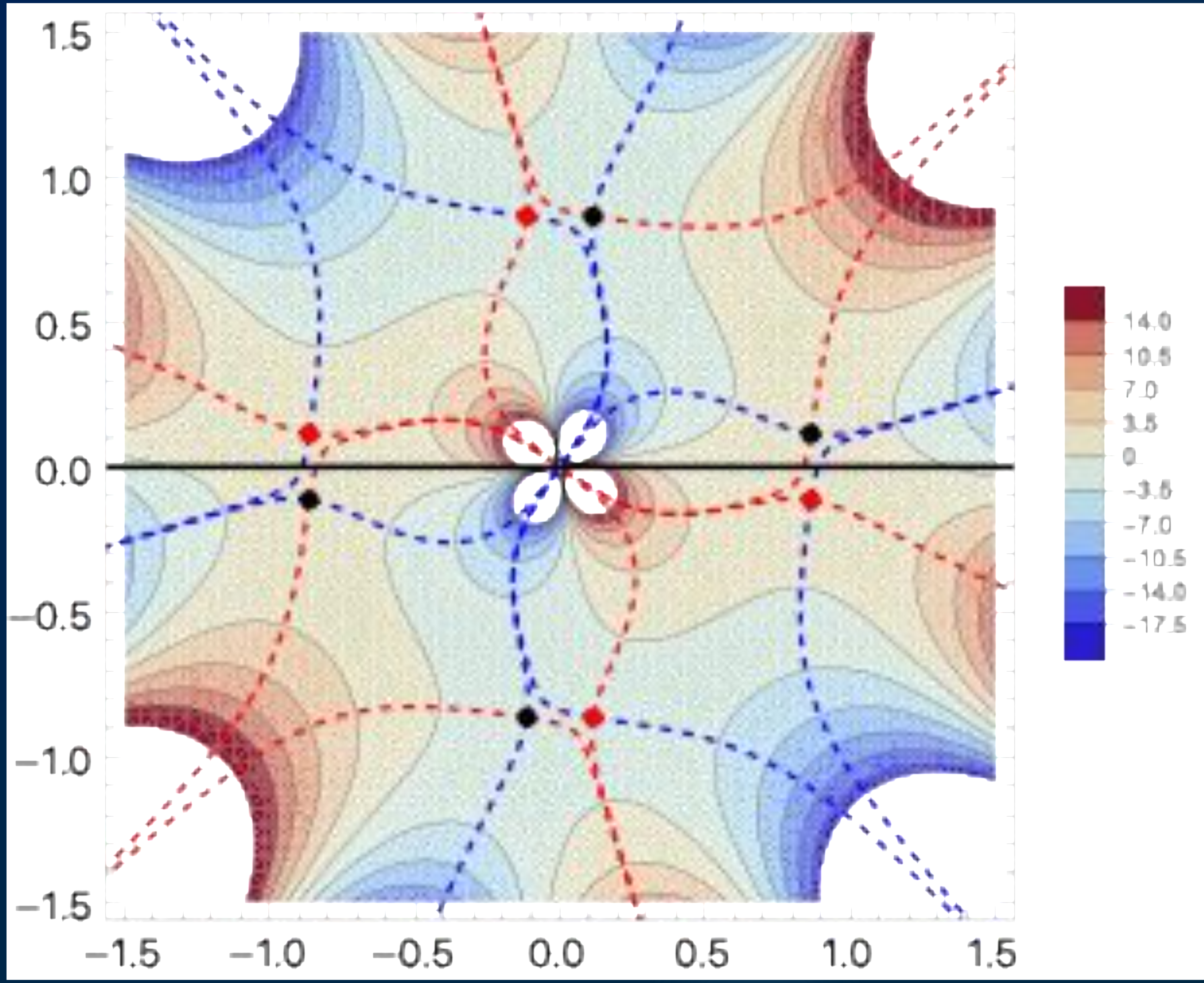
Picard-Lefschetz and Resurgence Analysis

$$\hbar = e^{i\theta}|\hbar| \quad \theta = +\frac{\pi}{10}$$

Only tunneling saddle points contribute

$$\theta = -\frac{\pi}{10}$$





Picard-Lefschetz and Resurgence Analysis

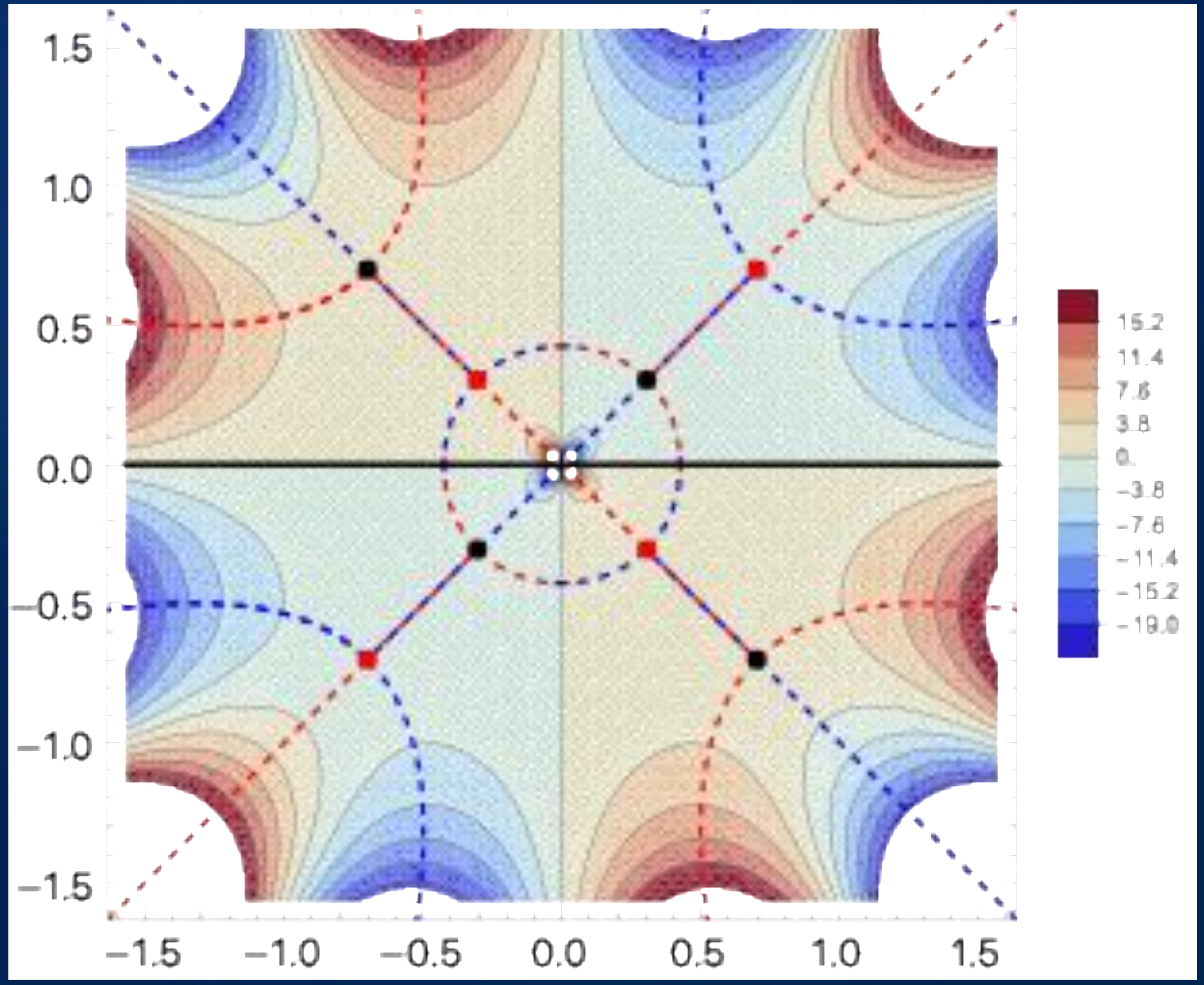
Stokes jumps -> Resurgence

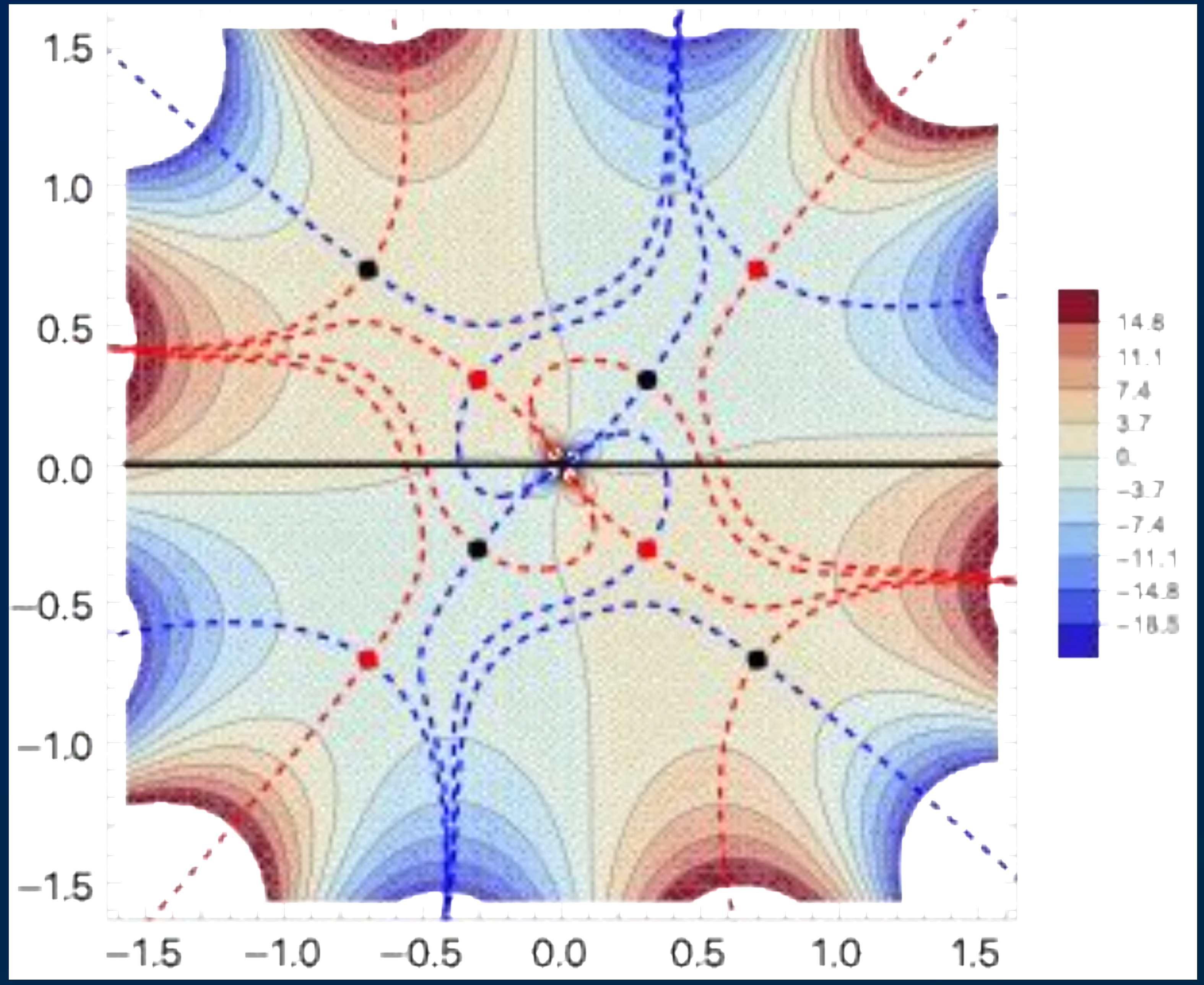
$$12\alpha\gamma + \beta^2 > 0, \quad \beta > 0$$

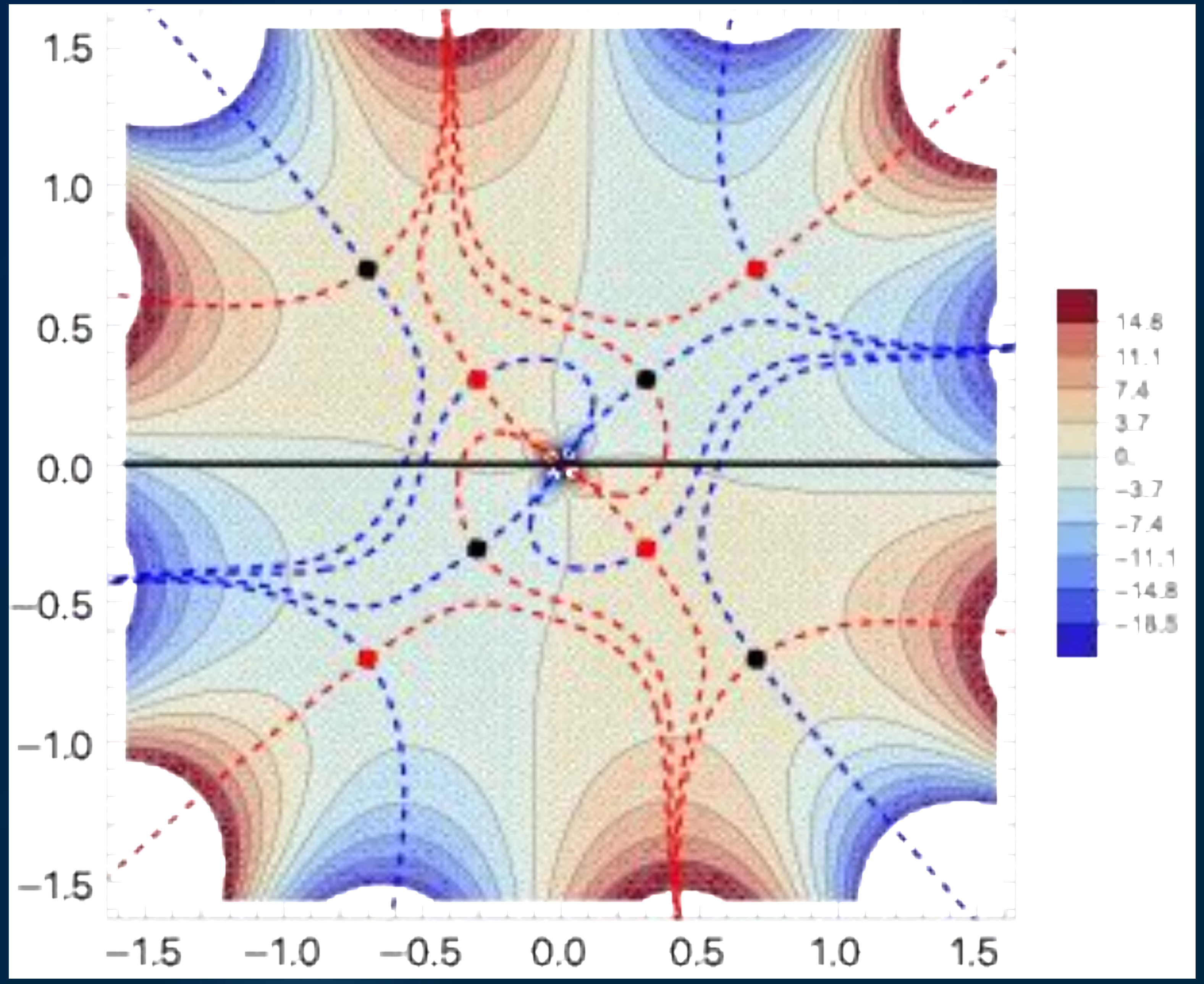
$$n_{x_s}\big|_{\theta\to 0_+} \neq n_{x_s}\big|_{\theta\to 0_-}$$

$\theta = +\frac{\pi}{10}$ Black and red saddles contribute

$\theta = -\frac{\pi}{10}$ Only black saddles contribute







Resurgence Analysis

[M. Honda, H. Matsui, K. Okabayashi, T. Terada, 2402.09981]

$$G(\hbar) = \sqrt{\frac{3iV_3}{4\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \exp\left[i\left(\alpha\hbar^2 x^6 + \beta x^2\right)\right] \qquad x = 0, \quad x_i = e^{\frac{i\arg\beta}{4}} e^{\frac{2n-1}{4}\pi i}\left(\frac{|\beta|}{3\alpha\hbar^2}\right)^{\frac{1}{4}} \quad (n = 1, ..., 4)$$

We expand it around x=0

Borel transformation

$$\mathrm{sign}(\beta) = \pm 1$$

$$G_0(\hbar) = \sum_{n=0}^{\infty} c_n \hbar^{2n} \qquad \mathcal{B}G_0(t) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(2n+1)} t^{2n} \cdots\cdots\rightarrow \mathcal{B}G_0(t) = e^{\pm\frac{i\pi}{4}} \sqrt{\frac{3iV_3}{4|\beta|}}\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{27t^2\alpha}{4\beta^3}\right)$$

Laplace transformation

$$\mathcal{S}_\theta G_0(\hbar) = \frac{1}{\hbar}\int_0^{\infty\cdot e^{i\theta}} \mathrm{d}t\, e^{-\frac{t}{\hbar}} \mathcal{B}G_0(t)$$

$$\left(\mathcal{S}_{0^+} - \mathcal{S}_{0^-}\right) G_0(\hbar) = -\frac{1}{\hbar}\sqrt{\frac{3V_3}{4\beta}} \int_{t_0}^{\infty} \mathrm{d}t\, e^{-\frac{t}{\hbar}}\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; 1 - \frac{27t^2\alpha}{4\beta^3}\right)$$

$$\left(\mathcal{S}_{0^+} - \mathcal{S}_{0^-}\right) G_0(\hbar) = -\sum_{x_s = x_1, x_3} \left(G(\hbar)\Big|_{x=x_s}^{\theta=0^+} - G(\hbar)\Big|_{x=x_s}^{\theta=0^-} \right)$$

Borel ambiguity is canceled by the ambiguity for the nontrivial saddle points

Perturbation Problem in Quantum Cosmology

Divergent perturbations

$$S_{\mathrm{GR}}[q,h,N] = S^{(0)}_{\mathrm{GR}}[q,N] + S^{(2)}_{\mathrm{GR}}[h,N] + \mathcal{O}(h^3)$$

Background part **Perturbation part**

[Phys.Rev.D 110 (2024) 2, 023503, Phys. Rev. D 107 (2023) 043511, Phys.Rev.D 97 (2018) 2, 023509 Phys. Rev. Lett. 119, 171301 (2017)]

$$S^{(2)}_{\mathrm{GR}}[h,N] = V_3 \int_0^1 N\mathrm{d}t \left\{ \frac{q^2}{8N^2}\dot{h}^2 - \frac{\alpha_{\mathrm{mode}}}{8} h^2 \right\} \qquad \alpha_{\mathrm{mode}} = ((n^2-3)+2)$$

Equation of Motion

$$\frac{\ddot{\chi}}{N^2} + \left[\frac{\alpha_{\mathrm{mode}}}{q^2} - \frac{1}{N^2}\frac{\ddot{q}}{q}\right]\chi = 0$$

Redefined field

$$\chi(t) = q(t)h(t)$$

General Solutions

$$\begin{aligned}
\chi(t) =& \sqrt{\left(H^2N^2t(t-1)+q_f t\right)\left(H^4N^2t(t-1)+H^2q_f t+\alpha_{\mathrm{mode}}\right)} \\
&\times \left\{ C_1 \left(\frac{H^2N^2(t-1)+q_f}{t}\right)^{\frac{\delta}{2}} \sqrt{\frac{H^2N^2(2t-1)+\sqrt{H^4N^4+N^2\left(-4\alpha_{\mathrm{mode}}-2H^2q_f\right)+q_f^2}+q_f}{H^2N^2(2t-1)-\sqrt{H^4N^4+N^2\left(-4\alpha_{\mathrm{mode}}-2H^2q_f\right)+q_f^2}+q_f}} \right. \\
&\left. + C_2 \left(\frac{t}{H^2N^2(t-1)+q_f}\right)^{\frac{\delta}{2}} \sqrt{\frac{H^2N^2(2t-1)-\sqrt{H^4N^4+N^2\left(-4\alpha_{\mathrm{mode}}-2H^2q_f\right)+q_f^2}+q_f}{H^2N^2(2t-1)+\sqrt{H^4N^4+N^2\left(-4\alpha_{\mathrm{mode}}-2H^2q_f\right)+q_f^2}+q_f}} \right\}
\end{aligned}$$

Divergent perturbations

Near the singularity the solution behaves as $\delta[N]=\frac{\sqrt{(H^2N^2-q_f)^2-4N^2\alpha_{\text{mode}}}}{(H^2N^2-q_f)}=-\sqrt{1-\frac{4N^2\alpha_{\text{mode}}}{(q_f-N^2H^2)^2}}$

$$\chi(t)\propto C_1\,F_1[N]t^{\frac{1}{2}(1-\delta)}+C_2\,F_2[N]t^{\frac{1}{2}(1+\delta)}\qquad(t\to 0)$$

$$S^{(2)}_{\text{on-shell}}[N]=\frac{\pi^2}{4}\left[q^2\frac{h\dot{h}}{N}\right]_0^1\ \propto C_1t^{-\delta},\ C_1C_2,\ C_2^2t^{\delta}$$

Zero

Divergent

$$S^{(2)}_{\text{on-shell}}[N]=-\frac{\pi^2q_fh_f^2\alpha_{\text{mode}}\left(-\left(H^2N^2-q_f\right)\delta[N]+H^2N^2+q_f\right)}{8N\left(\alpha_{\text{mode}}+H^2q_f\right)}$$

Divergent perturbations

Tunneling saddle point

$$N_{\mathrm{T}} = \frac{1}{H^2}\left[i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$$

Saddle point action

$$\frac{i}{\hbar} S^{(2)}_{\text{on-shell}}[N_{\mathrm{T}}] = +\frac{\pi^2 q_f h_f^2 \alpha_{\text{mode}}\left(\sqrt{\alpha_{\text{mode}}+1} - i\sqrt{q_f H^2 - 1}\right)}{4\hbar\left(q_f H^2 + \alpha_{\text{mode}}\right)}$$
$$\approx \frac{\pi^2 n(n^2-1)}{4\hbar H^2}\left[1 - i n^{-1} q_f^{1/2} H + \cdots\right] h_f^2$$

Inverse-Gaussian Wave function

$$\Psi(h) \sim e^{+\frac{\pi^2 \alpha_{\text{mode}}^{3/2}}{4\hbar H^2} h^2}$$

No-boundary saddle point

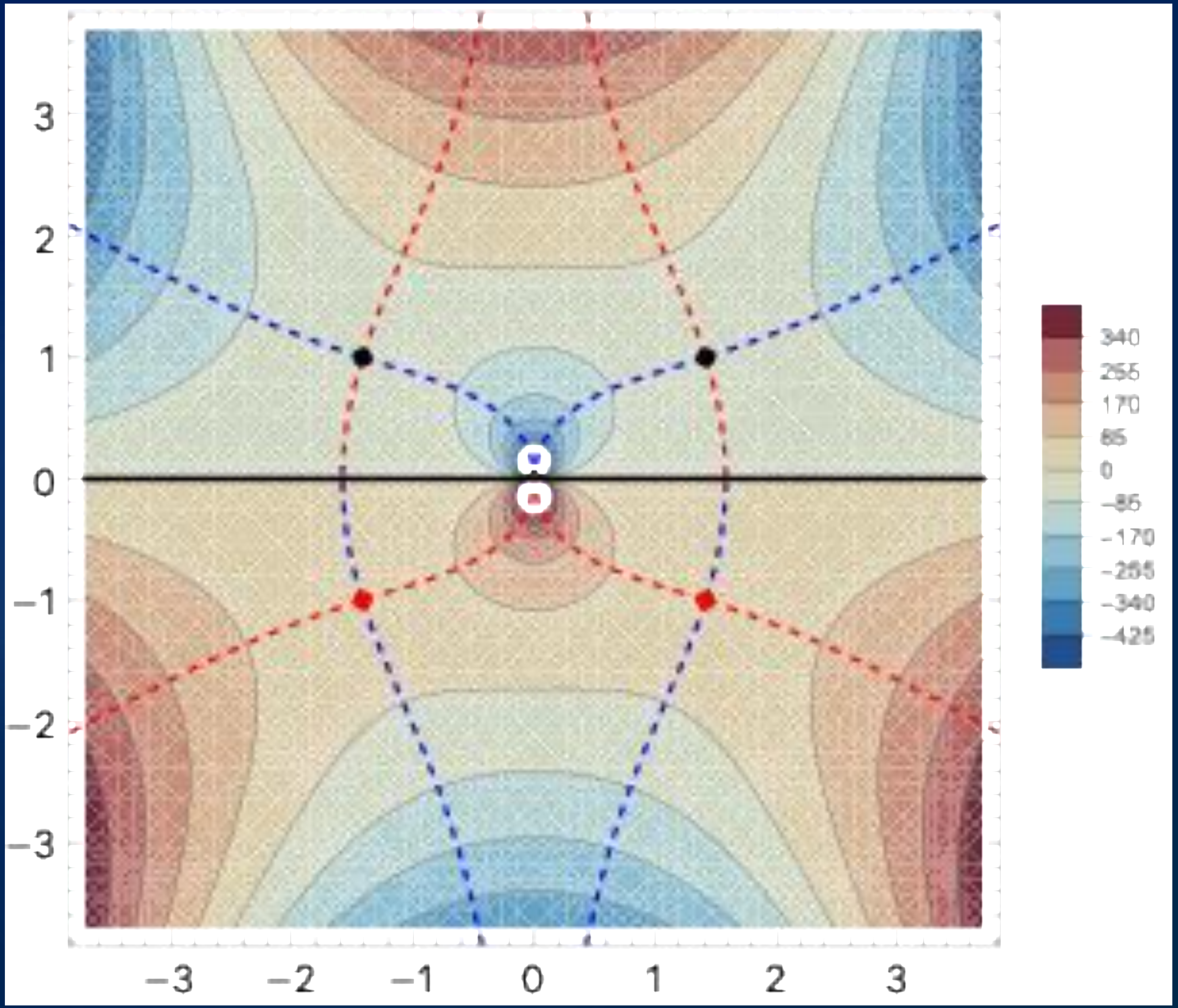
$$N_{\mathrm{HH}} = \frac{1}{H^2}\left[-i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$$

$$\Psi(h) \sim e^{-\frac{\pi^2 \alpha_{\text{mode}}^{3/2}}{4\hbar H^2} h^2}$$

[Phys.Rev.D 110 (2024) 2, 023503, Phys. Rev. D 107 (2023) 043511,
Phys.Rev.D 97 (2018) 2, 023509 Phys. Rev. Lett. 119, 171301 (2017)]

No Smooth Spacetime

[Phys.Rev.D 110 (2024) 2, 023503, Phys. Rev. D 107 (2023) 043511, Phys.Rev.D 97 (2018) 2, 023509 Phys. Rev. Lett. 119, 171301 (2017)]



Unstable Perturbation

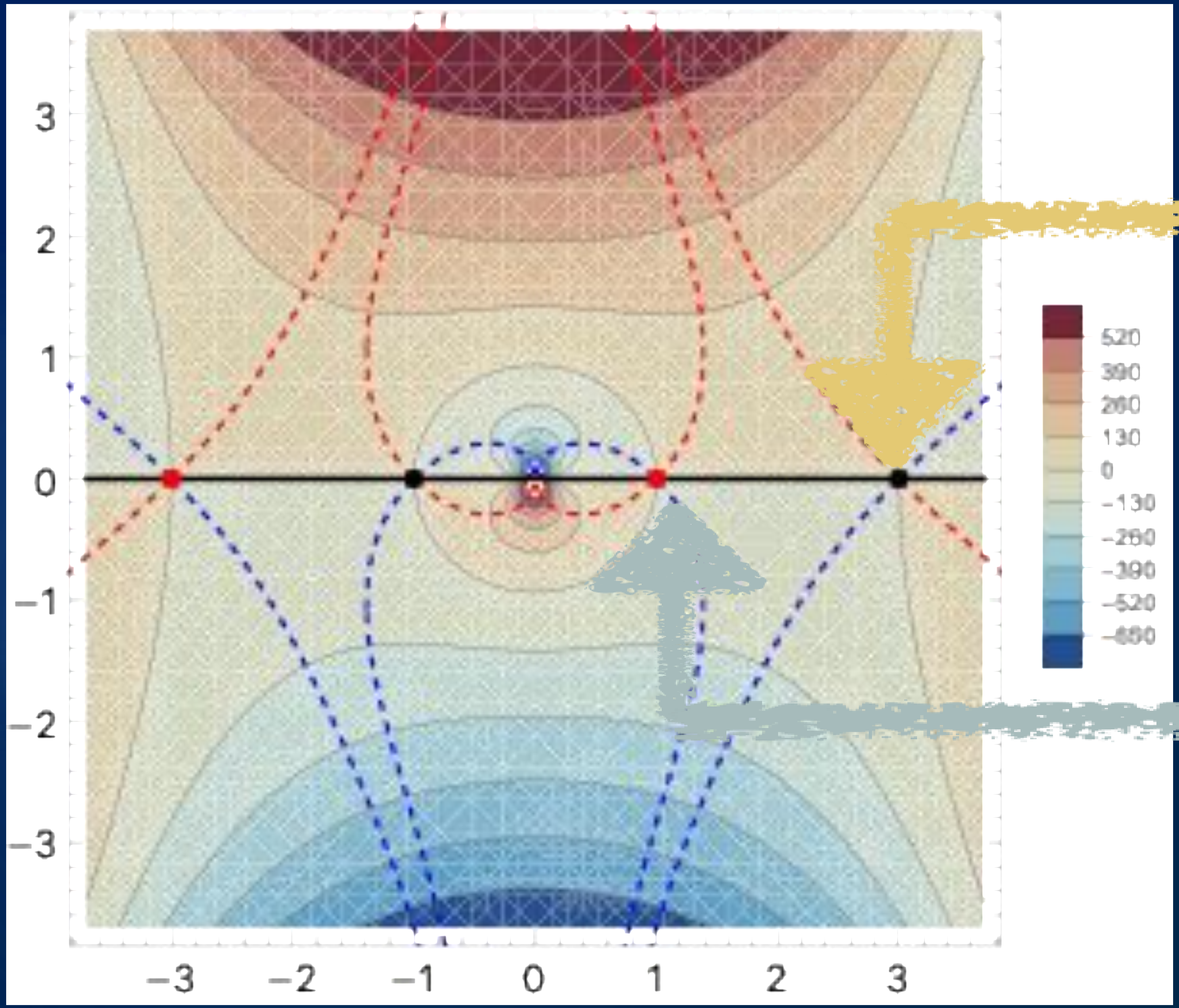
$$\Psi(h) \sim e^{+\frac{\pi^2 \alpha^{3/2}_{\text{mode}}}{4\hbar H^2} h^2}$$

Stable Perturbation

$$\Psi(h) \sim e^{-\frac{\pi^2 \alpha^{3/2}_{\text{mode}}}{4\hbar H^2} h^2}$$

Open or flat universe

[Phys.Rev.D 110 (2024) 2, 023503, Phys. Rev. D 100, 063517 (2019)]



Unstable Perturbation

$$\Psi(h) \sim e^{+\frac{\pi^2 \alpha_{\mathrm{mode}}^{3/2}}{4\hbar H^2} h^2}$$

Stable Perturbation

$$\Psi(h) \sim e^{-\frac{\pi^2 \alpha_{\mathrm{mode}}^{3/2}}{4\hbar H^2} h^2}$$

Trans-Planckian Physics

Trans-Planckian Physics

Modified dispersion relation

$$\omega^2 = \mathcal{F}\left(k_{\rm phys}\right) \quad k_{\rm phys} = \alpha_{\rm mode}^{\frac{1}{2}}/q^{\frac{1}{2}}$$

Physical momentum diverges at Big-bang singularity

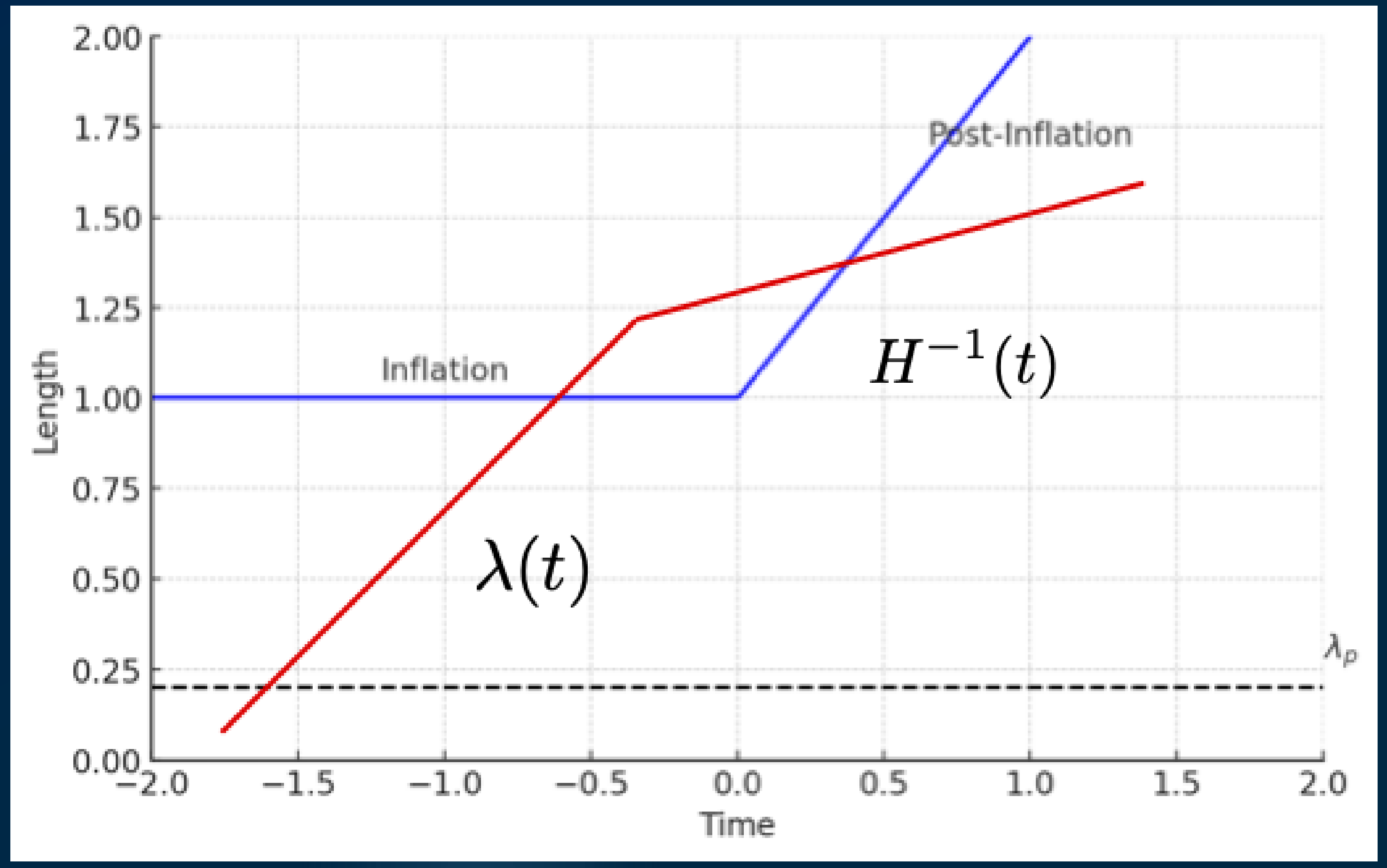
$$k_{\rm phys} \to \infty \qquad q \to 0$$

J. Martin and R. H. Brandenberger, Phys. Rev. D 63 (2001) 12350, Mod. Phys. Lett. A 16 (2001) 999, Phys. Rev. D 65 (2002) 103514, J. C. Niemeyer, Phys. Rev. D 63 (2001) 123502

Modified perturbative action

$$\chi(t) = q(t)h(t)$$

$$\begin{aligned} S^{(2)}[h,N] &= 2\pi^2 \int_0^1 N\mathrm{d}t \left[\frac{q^2}{8N^2}\dot{h}^2 - \frac{q}{8}\mathcal{F}(k_{\rm phys})h^2\right] + S_B^{(2)} \\ &= \frac{\pi^2}{4}\int_0^1 N\mathrm{d}t \left[\frac{1}{N^2}\left(\dot{\chi}^2 - 2\frac{\dot{\chi}\chi\dot{q}}{q} + \frac{\chi^2\dot{q}^2}{q^2}\right) - \mathcal{F}(k_{\rm phys})\frac{\chi^2}{q}\right] + S_B^{(2)} \end{aligned}$$



Modified EOM

$$\frac{1}{N}\partial_t\left(\frac{\dot{\chi}}{N}\right) + \left[\frac{\mathcal{F}(k_{\rm phys})}{q} - \frac{1}{qN}\partial_t\left(\frac{\dot{q}}{N}\right)\right]\chi = 0$$

Trans-Planckian Physics

1. Generalized Corley-Jacobson dispersion relation

$$\mathcal{F}\left(k_{\rm phys}\right) = k^2_{\rm phys} + k^2_{\rm phys} \sum_{j=1}^{p} b_j \left(\frac{k^2_{\rm phys}}{\mathcal{M}^2_{\rm UV}}\right)^j$$

S. Corley and T. Jacobson, Phys. Rev. D 54 (1996) 1568

introduced by higher-dimensional operators of gravity !

2. Trans-Planckian cutoff

$$\mathcal{F}\left(k_{\rm phys}\right) = \begin{cases} k^2_{\rm phys} & \text{for} \quad k^2_{\rm phys} \ll \mathcal{M}^2_{\rm UV} \\ \mathcal{M}^2_{\rm UV} & \text{for} \quad k^2_{\rm phys} \gg \mathcal{M}^2_{\rm UV}, \end{cases}$$

3. Unruh's dispersion relation

$$\mathcal{F}\left(k_{\rm phys}\right) = \mathcal{M}^2_{\rm UV} \tanh^{2/b}\left[\left(\frac{k^2_{\rm phys}}{\mathcal{M}^2_{\rm UV}}\right)^{\frac{b}{2}}\right]$$

W. G. Unruh, Phys. Rev. D 51 (1995) 2827

Generalized Corley-Jacobson dispersion relation

Equation of motion in UV (p=2)

$$\frac{\ddot{\chi}}{N^2} + \left\{ \frac{\alpha_{\text{mode}}}{q^2} \left[1 + b_2 \left(\frac{\alpha_{\text{mode}}}{q\mathcal{M}_{\text{UV}}^2} \right)^2 \right] - \frac{1}{N^2} \frac{\ddot{q}}{q} \right\} \chi = 0$$

Generalized Corley-Jacobson dispersion relation

$$\mathcal{F}\left(k_{\text{phys}}\right) = k_{\text{phys}}^2 + k_{\text{phys}}^2 \sum_{j=1}^{p} b_j \left(\frac{k_{\text{phys}}^2}{\mathcal{M}_{\text{UV}}^2} \right)^j$$

Solutions in UV (p=2)

$$\chi(t) = C_3 \left(N^2H^2(t-1)+q_f\right)^{\zeta_1} t^{\zeta_2} \exp\left[-\frac{\sqrt{-\alpha_{\text{mode}}\beta}\, N \left(N^2H^2(2t-1)+q_f\right)}{\left(N^2H^2-q_f\right)^2 \left(N^2H^2(t-1)+q_f\right) t} \right]$$
$$+ C_4 \left(N^2H^2(t-1)+q_f\right)^{\zeta_2} \tau^{\zeta_1} \exp\left[+\frac{\sqrt{-\alpha_{\text{mode}}\beta}\, N \left(N^2H^2(2t-1)+q_f\right)}{\left(N^2H^2-q_f\right)^2 \left(N^2H^2(t-1)+q_f\right) t} \right]$$

$$\beta = b_2 \alpha_{\text{mode}}^2 / \mathcal{M}_{\text{UV}}^4$$

$$\lambda = \sqrt{-\alpha_{\text{mode}}\beta}\, N / (H^2N^2 - q_f)^2$$

Near the singularity τ = 0 the solution behaves as

$$\chi(t) \propto C_3\, F_3[t,N] e^{-\frac{\lambda}{t}} + C_4\, F_4[t,N] e^{+\frac{\lambda}{t}}$$

$$\zeta_1 = 1 - 2\frac{N^3H^2\sqrt{-\alpha_{\text{mode}}\beta}}{\left(N^2H^2-q_f\right)^3}, \quad \zeta_2 = 1 + 2\frac{N^3H^2\sqrt{-\alpha_{\text{mode}}\beta}}{\left(N^2H^2-q_f\right)^3}$$

Divergent

Generalized Corley-Jacobson dispersion relation

[Phys.Rev.D 110 (2024) 2, 023503,
Phys. Rev. D 107 (2023) 043511]

Tunneling saddle point

$$N_{\mathrm{T}} = \frac{1}{H^2}\left[i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$$

Saddle point action

$$\frac{i}{\hbar} S^{(2)}_{\text{on-shell}}[N] = \begin{cases} +\frac{\pi^2}{4\hbar}\frac{\alpha_{\text{mode}}^{3/2} b_2^{1/2}}{\mathcal{M}_{\mathrm{UV}}^2} h_f^2 & \text{for} \quad \mathrm{Re}[\lambda] < 0 \\ -\frac{\pi^2}{4\hbar}\frac{\alpha_{\text{mode}}^{3/2} b_2^{1/2}}{\mathcal{M}_{\mathrm{UV}}^2} h_f^2 & \text{for} \quad \mathrm{Re}[\lambda] > 0\,. \end{cases}$$

$$\lambda[N_{\mathrm{T}}] = \frac{\sqrt{-\alpha_{\text{mode}}\beta N_{\mathrm{T}}}}{(N_{\mathrm{T}}^2 H^2 - q_f)^2} = -\frac{\alpha_{\text{mode}}^{\frac{3}{2}} b_2^{\frac{1}{2}}}{4 q_f \mathcal{M}_{\mathrm{UV}}^2}\left(1 + i\sqrt{q_f H^2 - 1}\right)$$

Inverse-Gaussian Wave function

$$\Psi(h_f) \sim e^{+\frac{\pi^2}{4\hbar}\frac{\alpha_{\text{mode}}^{3/2} b_2^{1/2}}{\mathcal{M}_{\mathrm{UV}}^2} h_f^2}$$

No-boundary saddle point

$$N_{\mathrm{HH}} = \frac{1}{H^2}\left[-i \pm \left(q_f H^2 - 1\right)^{1/2}\right]$$

$$\Psi(h_f) \sim e^{-\frac{\pi^2}{4\hbar}\frac{\alpha_{\text{mode}}^{3/2} b_2^{1/2}}{\mathcal{M}_{\mathrm{UV}}^2} h_f^2}$$

Trans-Planckian cutoff

[Phys.Rev.D 110 (2024) 2, 023503, Phys. Rev. D 107 (2023) 043511]

Equation of motion (UV)

$$\frac{\ddot{\chi}}{N^2} + \left\{ \frac{\mathcal{M}_{\rm UV}^2}{H^2N^2(t-1)t+q_f t} - \frac{2H^2}{H^2N^2(t-1)t+q_f t} \right\} \chi = 0$$

Trans-Planckian cutoff

$$\mathcal{F}\left(k_{\rm phys}\right) = \begin{cases} k_{\rm phys}^2 & \text{for} \quad k_{\rm phys}^2 \ll \mathcal{M}_{\rm UV}^2 \\ \mathcal{M}_{\rm UV}^2 & \text{for} \quad k_{\rm phys}^2 \gg \mathcal{M}_{\rm UV}^2 , \end{cases}$$

Solutions in UV

Meijer G-function

Divergent

$$\chi(t) = C_5\, G_{2,2}^{2,0}\left(\begin{matrix} \frac{3-\Delta/H}{2}, \frac{3+\Delta/H}{2} \\ 0, 1 \end{matrix} \middle| \frac{H^2N^2t}{H^2N^2 - q_1} \right)$$

$$\Delta = \sqrt{9H^2 - 4\mathcal{M}_{\rm UV}^2}$$

$$- C_6 \frac{H^2N^2t}{H^2N^2 - q_1}\, {}_2F_1\left(\frac{1-\Delta/H}{2}, \frac{1+\Delta/H}{2}; 2; \frac{H^2N^2t}{H^2N^2 - q_1} \right)$$

hypergeometric function

On-shell action

$$S^{(2)}_{\rm on-shell}[N] = \frac{\pi^2 N h_f^2 q_f}{8} \left\{ \frac{q_f\left(\mathcal{M}_{\rm UV}^2 - 2H^2\right)}{\left(H^2N^2 - q_f\right)} \frac{{}_2F_1\left(\frac{3-\Delta}{2}, \frac{3+\Delta}{2}; 3; \frac{H^2N^2}{H^2N^2-q_f}\right)}{{}_2F_1\left(\frac{1-\Delta}{2}, \frac{1+\Delta}{2}; 2; \frac{H^2N^2}{H^2N^2-q_f}\right)} - 2H^2 \right\}$$

Inverse-Gaussian Wave function

Summary

1. ローレンツ経路積分に基づいてHartle-Hawking 無境界仮説とトンネル仮説が近年定式化された。
2. Picard-Lefschetz理論(+Resurgence理論)を応用することで厳密に解析可能。
3. しかし、Background＋摂動で波動関数を解析すると時空の量子揺らぎは指数的に増大することが示唆される。
4. 量子宇宙論の摂動論的問題は一般相対性理論を超えた枠組み（Trans-Planckian Physics）においても存在する。