Φ^4型の行列模型と調和振動子やカロジェロ型の可積分系との関係

Relationship between Φ^4 matrix model & N-body harmonic oscillator or Calogero-Moser model

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arXiv: 2308.11523, 2311.10974 (Lett. in Math. Phys.)

離散的手法による場と時空のダイナミクス'24 @ 東工大

9月5日 (9/2 ~ 9/5)

Talk Plan

1 History and Overview

2 Matrix model from Non-Com. Quantum Field Theory

3. Φ^4 model $\Rightarrow$ Harmonic Oscillator System

Calogelo - Moser Model

4. Viraso (Witt) alg

5. Loop - eq.

6. Connected Green fun.

7. Summary

§1 History

▶ 90s' Matrix model

◎ 2D gravity ⇔ random matrix

Brezin - Kazakov, Gross - Migdal, etc.

Kontsevich model (Witten conjecture)

Fukuma - Kawai
Nakayama

$$Z[J] = \int \mathcal{D}\Phi \, \exp\left(-\mathrm{tr}\left(\Lambda \Phi^2 + \Phi^3\right)\right)$$

Φ : Hermite matrix, $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \cdots)$

Makeenko - Semenoff solve this in $N \to \infty$

Matsumoto's talk $t_n = -(2n-1)!! \sum_{l} \frac{1}{(\lambda_l)^{2n+1}}$

▶ 2000's QFT on N.C. space

N.C. field Theory $\Rightarrow$ Matrix model.

Grosse-Steinacker ('05, '06)

Φ^3 model (basically Kontsevich model) Renormalizable

Grosse-Wulkenhaar ('04)

Φ^4 models in 2, 4 dim are Renormalizable

Φ^4 model is solvable

(SD-eq is recursivly determined.)

§2. ~The Origin from N.C. field Theory~

$\mathbb{R}^2_\theta$: Moyal plane $\quad [A,B] := AB - BA$

$$[x^1, x^2] = i\underbrace{\theta}_{\text{N.C. parameter}} \Leftrightarrow [z, \bar{z}] = 2\theta$$

- Annihilation $\left(a := \dfrac{z}{\sqrt{2\theta}} \right)$ Creation $\left(a^\dagger := \dfrac{\bar{z}}{\sqrt{2\theta}} \right)$ op.

$$\Rightarrow [a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0$$

- $\dfrac{\partial}{\partial z} = -\dfrac{1}{\sqrt{2\theta}}[a^\dagger, \;\;], \quad \dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{\sqrt{2\theta}}[a, \;\;]$

- Fock sp. $[a, a^\dagger] = 1$, $[a, a] = [a^\dagger, a^\dagger] = 0$

$|0\rangle$: $a|0\rangle = 0$, $|n\rangle := \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle$

Number op. $N := a^\dagger a$ $\quad N|n\rangle = n|n\rangle$

$\langle n|$ = dual of $|n\rangle$

$\langle n|m\rangle = \delta_{nm}$

- Scalar field $\Phi = \underline{\sum \Phi_{nm} |m\rangle\langle n|}$
  $(C^\infty(\mathbb{R}^d))$

Hermitian Matrix !!

$\int d^2x \rightarrow \theta^2 \, \mathrm{Tr}$

Action

$$S_1 = \int d^4x \, -\phi \left(\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \right) \phi$$

$$= \frac{\theta^2}{2\theta} \mathrm{Tr}\, \phi \, [a^\dagger, [a, \phi]] \qquad N = a^\dagger a$$

$$= \theta \; \mathrm{Tr} (\phi N \phi - \underline{\theta a^\dagger \phi a \phi})$$

Removing this term by a counter Lagrangian

Renormalizable model is obtained.

$$S_m = \theta \, \mathrm{Tr} \frac{\mu^2}{2} \phi^2$$

μ: const. (mass)

ex). Φ^3 model $\quad \Phi$: $N\times N$ Hermitian matrix

$$S = N\,\mathrm{tr}\left(E\Phi^2 - A\Phi + \frac{2}{3}\Phi^3\right)$$ $\leftrightarrow$ { Kontsevich model, KdV hierarchy }

$$E_{ij} = \left(\frac{1}{2}\mu^2 + i\right)\delta_{ij}, \quad A : \text{const}.$$

$$Z[J] = \int D\Phi\, e^{-S + N\,\mathrm{tr}(J\Phi)}$$

$$\left(D\Phi = \prod_i d\Phi_{ii} \prod_{i<j} d\Phi_{ij}^{Re}\, d\Phi_{ij}^{Im} \right)$$

$$\log \frac{Z[J]}{Z[0]} = \sum_{B=1}^{\infty} \sum_{1\le N_1\le\cdots\le N_B}^{\infty} \sum_{p_i^j=0}^{N} N^{2-B} \frac{G_{|p_1^1\cdots p_{N_1}^1|\cdots|p_1^B\cdots p_{N_B}^B|}}{S(N_1,\cdots,N_B)}$$

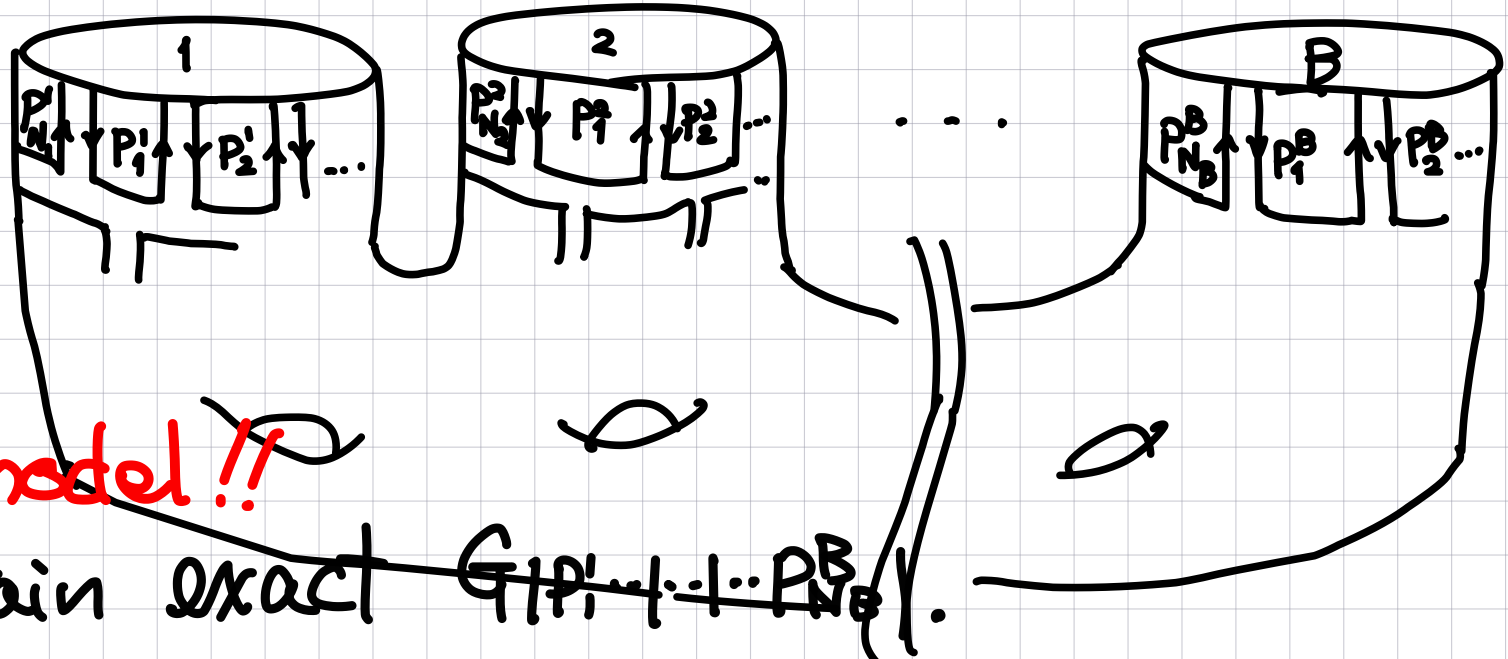
$$\times \prod_{\beta=1}^{B} \frac{\prod_{k=1}^{N_\beta} J_{p_k p_{k+1}}}{N_\beta}$$

statistical factor

$N_1+N_2+\cdots+N_B$ pts connected Green fun.

$$G|p^1_1 p^1_2 \cdots p^1_{N_1}|p^2_1 p^2_2 \cdots p^2_{N_2}| \cdots |p^B_1 p^B_2 \cdots p^B_{N_B}|$$

$$= \sum$$



Solvable model!!

We can obtain exact $G|p^1_1 \cdots | \cdots | \cdots p^B_{N_B}|$.

H. Grosse - A.S. - R. Wulkenhaar ('16, '17.) for $N \to \infty$

N. Kanomata - A.S. ('23) for finite N

Φ^4 model is also solvable

H. Grosse - R. Wulkenhaar - A. Hock ('19, '21)

J. Branahl - A. Hock - R. Wulkenhaar ('22)

↑ topological recursion

Hermitian matrix

§3 Φ^4 model ⇒ Harmonic Oscillator System

$$S = N\,\mathrm{tr}\left(E\Phi^2 + \frac{\eta}{4}\Phi^4\right) \qquad \eta \in \mathbb{R}_{>0}$$

$$Z = \int d\Phi\, e^{-S} \qquad E = \mathrm{diag}(E_1, E_2, \cdots, E_N)$$

$d\Phi = \prod_i d\Phi_{ii} \prod_{i<j} d\Phi_{ij}^{Re}\, d\Phi_{ij}^{Im}$

$E_i > 0, \quad E_i \neq E_j$

Main Thm. 1

$$\Psi(E,\eta) := e^{-\frac{N}{2\eta}\sum_i E_i^2}\,\Delta(E)\,Z,$$

where $\Delta(E) = \prod_{k<l}(E_l - E_k)$ Vandermonde

$$\Rightarrow \mathcal{H}_{HO}\Psi = 0, \qquad \text{Schrödinger eq}$$

$$\mathcal{H}_{HO} = -\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2 \ : \text{Hamiltonian}$$

N-body (N-dim) Harmonic Oscillator System

Proof). $E = \begin{pmatrix} E_1 & & & O \\ & E_2 & & \\ & & \ddots & \\ O & & & E_N \end{pmatrix} = U H U^\dagger$ $\quad H = (H_{ij})$

Hermite non-diagonal

$$S = N \operatorname{tr}\left(H\Phi^2 + \frac{\eta}{4}\Phi^4\right), \quad Z = \int d\Phi\, e^{-S}$$

S-D eq $\displaystyle \int d\Phi \frac{\partial}{\partial \Phi_{ij}}\left(\Phi_{ij} e^{-S}\right) = 0$

$$\Leftrightarrow Z - N\sum_k \left(H_{ki}\langle \Phi_{ij}\Phi_{jk}\rangle + H_{jk}\langle \Phi_{ki}\Phi_{ij}\rangle\right) - N\eta \sum_{k,\ell}\langle \Phi_{jk}\Phi_{k\ell}\Phi_{\ell i}\Phi_{ij}\rangle = 0$$

, where $\langle \mathcal{O}\rangle = \int d\Phi\, \mathcal{O}\, e^{-S}$

$$\frac{\partial Z}{\partial H_{ij}} = -N\sum_k \langle \Phi_{jk}\Phi_{ki}\rangle, \quad \frac{\partial^2 Z}{\partial H_{ij}\partial H_{mn}} = N^2\sum_{k,\ell}\langle \Phi_{jk}\Phi_{k\ell}\Phi_{n\ell}\Phi_{\ell m}\rangle$$

Diff eq.

$$\left(N^2 + 2\sum_{i,k} H_{ki}\frac{\partial}{\partial H_{ki}} - \frac{\eta}{N}\sum_{i,k}\left(\frac{\partial}{\partial H_{ki}}\frac{\partial}{\partial H_{ik}}\right)\right) Z = 0$$

@ Change of valiables

Z depends on only E_i.

$$\mathbb{R}^{N^2} \to \mathbb{R}^{N} \times U(N) \qquad H_{ij} \mapsto E_i + \cancel{\text{others}}$$

- Laplacian: $\displaystyle\sum_{i,j} \frac{\partial}{\partial H_{ji}}\frac{\partial}{\partial H_{ij}} = \sum_i \left(\frac{\partial}{\partial E_i}\right)^2 + \sum_{i\neq j}\frac{1}{E_i-E_j}\left(\frac{\partial}{\partial E_i}-\frac{\partial}{\partial E_j}\right)$
- $\displaystyle\sum_{i,j} H_{ij}\frac{\partial}{\partial H_{ij}} = \sum_k E_k \frac{\partial}{\partial E_k}$

$$\underbrace{\left\{\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{\eta}{N}\sum_{i\neq j}\frac{1}{E_i-E_j}\left(\frac{\partial}{\partial E_i}-\frac{\partial}{\partial E_j}\right) - 2\sum_k E_k\frac{\partial}{\partial E_k} - N^2\right\}}_{=:\ \mathcal{L}_{SD}} Z = 0$$

Lem. Diagonalization

N- Body harmonic oscillator Hamiltonian

$$\mathcal{H}_{HO} := -\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2 ,$$

$$e^{-\frac{N}{2\eta}\sum_i E_i^2}\,\Delta(E)\,\mathcal{L}_{SD}\,\Delta(E)^{-1}\,e^{\frac{N}{2\eta}\sum_i E_i^2} = -\mathcal{H}_{HO}$$

$\Downarrow$

Thm.

$$\Psi(E,\eta) := e^{-\frac{N}{2\eta}\sum_i E_i^2}\,\Delta(E)\,Z(E,\eta)$$

$$\Rightarrow \quad \mathcal{H}_{HO}\,\Psi(E,\eta) = 0$$

Ψ is a 0-energy solution of HO system //

▷ Concrete expression

↙ IZ integral

$$Z(E,\eta)=\frac{c}{\Delta(E)}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}dx_i\, e^{-N\left(\frac{\eta}{4}x_i^4+E_ix_i^2\right)}\right)\left(\prod_{k<l}\frac{x_k-x_l}{x_k+x_l}\right)$$

$$\Updownarrow$$

$$\Psi(E,\eta)=c\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}dx_i\, e^{-N\left(\frac{\eta}{4}x_i^4+E_ix_i^2+\frac{1}{2\eta}E_i^2\right)}\right)\left(\prod_{k<l}\frac{x_k-x_l}{x_k+x_l}\right)$$

⇓ Bruijn's formula

$$Z(E,\eta)=\frac{c}{\Delta(E)}\operatorname*{Pf}_{i,j}M_{ij},\quad \Psi=c\,e^{-\frac{N}{2\eta}\sum_i E_i^2}\operatorname*{Pf}_{i,j}M_{ij}$$

where

$$M_{i,j}=\int_{\mathbb{R}^2}dx\,dy\left(\frac{x-y}{x+y}\right)e^{-2N\left(V(x)+E_ix^2+V(y)+E_jy^2\right)}$$

$$V(x)=\frac{\eta}{4}x^4$$

Note $\mathcal{H}_{HO} = \sum_{i=1}^{N}\left\{-\left(\frac{\partial}{\partial u_i}\right)^2 + u_i^2\right\} > 0$ possitive

In $L^2(\mathbb{R}^N)$ there is no solution of

$$\mathcal{H}_{HO}\,\Psi = 0 \quad \text{without} \quad \Psi = 0 .$$

What's happen ?

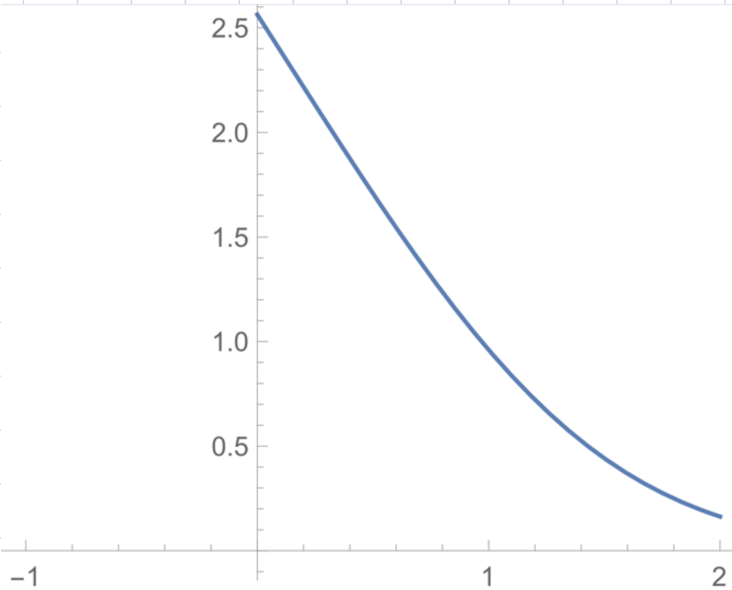
Rem. $N=1$ case

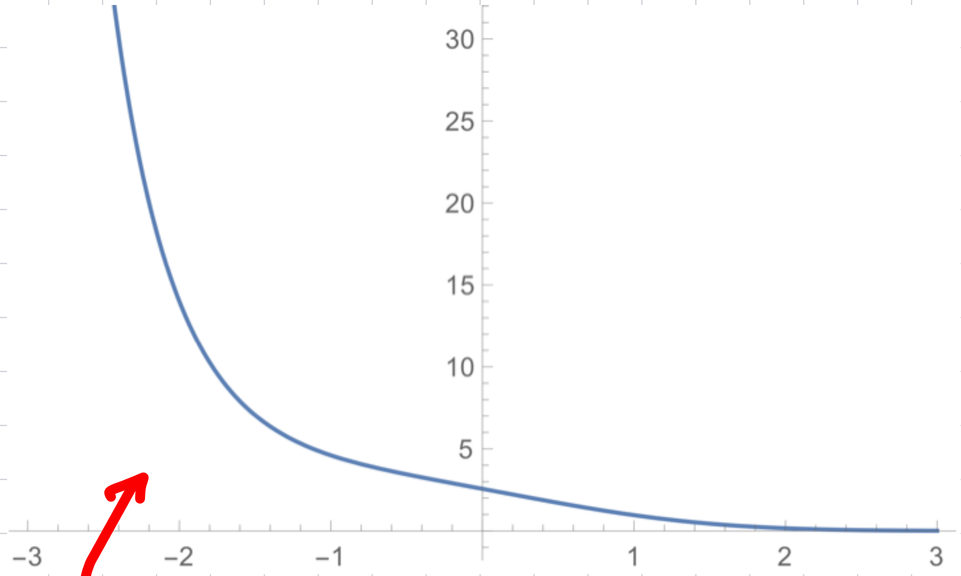
H-O (Weber eq) $\quad y''(u) = u^2\, y(u) \quad \left(u = \sqrt{\tfrac{N}{\eta}}\,E\right)$

$$\Psi(u) = e^{-\frac{u^2}{2}} \int_{-\infty}^{\infty} dx\, e^{-\sqrt{\eta}\,u x^2 - \frac{\eta}{4} x^4} = \frac{1}{\eta^{\frac{1}{4}}} \sqrt{u}\, K_{\frac{1}{4}}\!\left(\frac{u^2}{2}\right)$$

modified Bessel fun of 2nd kind

$$\sqrt{u}\, K_{\frac{1}{4}}\left(\frac{u^2}{2}\right) \qquad\qquad e^{-\frac{u^2}{2}}\int_{-\infty}^{\infty} dx\, e^{-u x^2 - \frac{1}{4}x^4}$$





Not $L^2(\mathbb{R})$ function

For $N > 1$, the nature of

$$\Psi(E,\eta) = C \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} dx_i\, e^{-N\left(\frac{\eta}{4}x_i^4 + E_i x_i^2 + \frac{1}{2\eta}E_i^2\right)} \right) \left(\prod_{k<l} \frac{x_k - x_l}{x_k + x_l} \right)$$

$\left(= C e^{-\frac{N}{2\eta}\sum_i E_i^2}\, \underset{i,j}{\mathrm{pf}}\, M_{ij} \right)$ is still unknown.

Hermitian matrix

§3 Φ^4 model $\Rightarrow$ Harmonic Oscillator System

$$S = N\,\mathrm{tr}\left(E\Phi^2 + \frac{\eta}{4}\Phi^4\right) \qquad \eta \in \mathbb{R}_{>0}$$

$$Z = \int d\Phi \; e^{-S} \qquad E = \mathrm{diag}(E_1, E_2, \cdots, E_N)$$

$$d\Phi = \prod_i d\Phi_{ii} \prod_{i<j} d\Phi_{ij}^{Re}\, d\Phi_{ij}^{Im} \qquad E_i > 0, \quad E_i \neq E_j$$

Main Thm. 1

$$\Psi(E,\eta) := e^{-\frac{N}{2\eta}\sum_i E_i^2}\,\Delta(E)\, Z,$$

where $\Delta(E) = \prod_{k<l}(E_l - E_k)$ Vandermonde

$$\Rightarrow \mathcal{H}_{HO}\Psi = 0, \qquad \text{Schrödinger eq}$$

$$\mathcal{H}_{HO} = -\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2 \quad : \text{Hamiltonian}$$

N-body (N-dim) Harmonic Oscillator System

▶ Hermitian matrix ⇒ Real Sym matrix

Φ^4 model ⇒ Calogero-Moser System

$$S = N\,\mathrm{tr}\left(E\Phi^2 + \frac{\eta}{4}\Phi^4\right) \qquad \eta \in \mathbb{R}_{>0}$$

$$Z = \int d\Phi\, e^{-S} \qquad E = \mathrm{diag}(E_1, E_2, \cdots, E_N)$$

$d\Phi = \prod_i d\Phi_{ii} \prod_{i<j} \Phi_{ij}$ $\qquad E_i > 0,\quad E_i \neq E_j$

Main Thm. 2

$$\Psi(E,\eta) := e^{-\frac{N}{\eta}\sum_i E_i^2}\, \Delta^{\frac{1}{2}}(E)\, Z,$$

where $\Delta(E) = \prod_{k<l}(E_l - E_k)$ Vandermonde

$$\Rightarrow \mathcal{H}_C \Psi = 0, \qquad \text{Schrödinger eq}$$

$$\mathcal{H}_C = -\frac{\eta}{2N}\left(\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{1}{4}\sum_{i\neq j}\frac{1}{(E_i - E_j)^2}\right) + \frac{2N}{\eta}\sum_{i=1}^{N}(E_i)^2 : \text{Hamiltonian}$$

Calogero-Moser model

§4 Virasoro (Witt) alg. for Hermitian Φ

$$\mathcal{H}_{HO} = -\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2 \quad : \text{Hamiltonian}$$

$$\downarrow \quad y_i := \sqrt{\frac{N}{\eta}}\, E_i$$

$$\mathcal{H}_{HO} = \sum_i \left(-\left(\frac{\partial}{\partial y_i}\right)^2 + y_i\right)^2$$

$$= \frac{1}{4}\sum_i \{a_i, a_i^\dagger\}$$

$\{A, B\} := AB + BA$

Creation, anihilation

$$\begin{cases} a_i^\dagger := \frac{1}{\sqrt{2}}(y_i - \partial_i), \\ a_i := \frac{1}{\sqrt{2}}(y_i + \partial_i) \\ [a_i, a_j^\dagger] = \delta_{ij} \end{cases}$$

▷ Virasoro generator

$$L_{-n} := \sum_{i=1}^{N}\{\alpha (a_i^\dagger)^{n+1} a_i + (1-\alpha) a_i (a_i^\dagger)^{n+1}\} \qquad \alpha \in \mathbb{R}$$

$$[L_n, L_m] = (n-m) L_{n+m}$$

$$L_0 = \frac{1}{2}\mathcal{H}_{HO} + \frac{N}{2} - \alpha N$$

i.e.) $[\frac{1}{2}\mathcal{H}_{HO}, L_{-m}] = m L_{-m}$

$$\Downarrow \quad \begin{cases} g := e^{\frac{N}{2n}\sum_i E_i^2} \Delta^{-1}(E) \\ \tilde{L}_n := g L_n g^{-1}, \end{cases} \quad \mathcal{L}_{SD} := -g\mathcal{H}_{HO} g^{-1}$$

$$[\mathcal{L}_{SD}, \tilde{L}_{-m}] = -2m \tilde{L}_{-m}$$

Thm3.

$$\boxed{\mathcal{L}_{SD}(\tilde{L}_{-m} Z(E,\eta)) = -2m(\tilde{L}_{-m} Z(E,\eta))}$$

↑ The same thm is obtained for Real Sym Φ.

for real sym Matrix model

▷ Bergshoeff and Vasiliev ('95)

Calogero-Moser model

$$H_C = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial y_i^2} + y_i^2\right) + \sum_{j>k}\frac{\beta(\beta-1)}{(y_j - y_k)^2}$$

$\left(y_i = \sqrt{\frac{2N}{g}}\, E_i \ , \ \beta = \frac{1}{2} \right.$ for the real Φ^4 model $\left.\right)$

- creation anhilation op.

$$a_i = \frac{1}{\sqrt{2}}(y_i + D_i) \ , \quad a_i^\dagger = \frac{1}{\sqrt{2}}(y_i - D_i)$$

where $D_i = \frac{\partial}{\partial y_i} + \beta \sum_{j \neq i} \frac{1 - K_{ij}}{y_i - y_j}$, Dunkl op.

K_{ij} : permutation op $\quad K_{ij}\, y_j = y_i$

$$\Rightarrow [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \ , \quad [a_i, a_j^\dagger] = \delta_{ij}\left(1 + \beta\sum_{l=1}^{N} K_{il}\right) - \beta K_{ij}$$

$$L_{-n} := \sum_{i=1}^{N} \left(\alpha (a_i^\dagger)^{n+1} a_i + (1-\alpha)\, a_i (a_i^\dagger)^{n+1} \right)$$

$\Rightarrow$ Virasoro (Witt) alg. $[L_n, L_m] = (n-m) L_{n+m}$

$$H := \frac{1}{2} \sum_{i}^{N} \{ a_i, a_i^\dagger \}$$

$$= L_0 - \left(\frac{1}{2} - \alpha \right) N + \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \sum_{i \neq j} K_{ij}$$

$\mathrm{Rest}(H)$: Restriction to K_{ij} inv sp

$$\Rightarrow \mathrm{Rest}(H) = \prod_{j>k} (y_j - y_k)^{-\frac{1}{2}} H_c \prod_{j>k} (y_j - y_k)^{\frac{1}{2}}$$

$$= -\frac{1}{2} e^{-\sum_i y_i^2} \mathcal{L}_{SD}\, e^{\sum_i y_i^2}$$

S-D op for real Sym Φ^4 matrix model

Note $\quad K_{ij} Z = Z$

$$K_{ij}\, e^{\sum_i y_i^2} = e^{\sum_i y_i^2}, \cdots$$

$\Rightarrow$ We can ignore "Rest"

$\Rightarrow$ The following can be discussed in the same way as in the case of H-O.

$$\tilde{L}_{-m} := e^{\frac{1}{2}\sum_i y_i^2} L_{-m}\, e^{-\frac{1}{2}\sum_i y_i^2}$$

$$[\mathcal{L}_{SD}, \tilde{L}_{-m}] = -2m\, \tilde{L}_{-m}$$

<u>Thm3.</u>

$$\boxed{\mathcal{L}_{SD}\left(\tilde{L}_{-m} Z(E,\eta)\right) = -2m\left(\tilde{L}_{-m} Z(E,\eta)\right)}$$

Rem.

Awata-Matsuo-Odake-Shiraishi '95

Hermitian Matrix model

$$V(\Phi) = \sum_{n}^{\infty} g_n \Phi^n \quad \leftarrow \text{different theory}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{different Virasoro constraint.}}}{\mathcal{L}_m} Z = 0$$

§7 Summary

Typical quantum integrable

@ Schrödinger eq for 0-energy $\mathcal{H}\Psi = 0$

$$\mathcal{H} = -\frac{\hbar}{2N}\left(\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{1}{4}\sum_{i\neq j}\frac{1}{(E_i-E_j)^2}\right) + \frac{2N}{\hbar}\sum_{i=1}^{N}(E_i)^2 \quad : \text{Hamiltonian}$$

Calogero - Moser model or Harmonic Oscillators

$$\Psi = e^{-\frac{2N}{2\hbar}\sum_i E_i^2}\,\Delta(E)^{\frac{1}{2}}\,Z$$

$$\mathcal{H} = \frac{1}{2}\sum_i \{a_i, a_i^\dagger\}$$

@ G-W type Φ^4 (Hermitian / Real Sym) matrix model

$$Z = \int d\Phi\, e^{-S},$$

$$S = N\,\mathrm{tr}\left\{E\Phi^2 + \frac{\hbar}{4}\Phi^4\right\}$$

S-Deq $\mathcal{L}_{SD} Z = 0$

gauge trans

@ Virasoro alg.

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m}$$

$$\mathcal{L}_{SD}(\tilde{L}_{-m} Z) = -2m\tilde{L}_{-m} Z$$

Thank you very much for your kind attention!